M408N Final Exam, December 13, 2011

1) (32 points, 2 pages) Compute $dy/dx$ in each of these situations. You do not need to simplify:

a) $y = x^3 + 2x^2 - 14x + 32$

b) $y = (x^3 + 7)^5$.

c) $y = \frac{\sin(x)}{x^2+1}$

d) $y = \ln(\sin(x^2))$

e) $y = e^x \tan^{-1}(x)$

f) $y = x^{\sin(x)}$

g) $xy + e^x + \ln(y) = 17$. (For this part, you can leave your answer in terms of both $x$ and $y$)

h) $y = \int_3^{2x} \sin(t^2)dt$.

2) (8 points) Some values of the function differentiable $f(x)$ are listed in the following table.

| $x$ | $f(x)$ |
|---|---|
| 2.95 | 8.8050 |
| 2.96 | 8.8432 |
| 2.97 | 8.8818 |
| 2.98 | 8.9208 |
| 2.99 | 8.9602 |
| 3.00 | 9.0000 |
| 3.01 | 9.0402 |
| 3.02 | 9.0808 |
| 3.03 | 9.1218 |
| 3.04 | 9.1632 |
| 3.05 | 9.2050 |

a) Compute the average rate of change between $x = 2.99$ and $x = 3.04$.

b) Estimate, as accurately as you can, the value of $f'(3)$.

3) (10 points) Let $f(x) = 10x^3 - 74$.

a) Find the equation of the line tangent to the curve $y = f(x)$ at (2,6).

b) Use this tangent line (or equivalently, a linear approximation) to estimate $f(2.1)$.

c) Use this tangent line (or equivalently, a linear approximation) to estimate a value of $x$ for which $f(x) = 0$. (Congratulations. You just computed the cube root of 7.4 by hand.)

4) (12 points, 2 pages!) The position of a particle is $f(t) = t^4 - 6t^2 + 8$. (Note that this factors as $(t^2 - 2)(t^2 - 4)$. Note also that $t$ can be positive or negative; the domain of the function is the entire real line.)

a) Make a sign chart for $f$, indicating the values of $t$ where $f(t)$ is positive, where $f(t)$ is negative, and where $f(t) = 0$.

b) At what times is the particle moving forwards? (Either express your answer in interval notation or make a relevant sign chart.)

c) At what times is the velocity increasing?

d) Sketch the graph $y = f(t)$. Mark carefully the local maxima, the local minima, and the points of inflection.

5) (8 points) Consider the function

$$f(x) = \begin{cases} 1 + x & x \leq 0 \\ e^x & x > 0. \end{cases}$$

a) Compute

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x}.$$

b) Compute

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x}.$$

c) Is $f$ differentiable at $x = 0$? Explain why or why not. If $f$ is differentiable, compute $f'(0)$.

6) (6 points) Consider the expression

$$\lim_{N\to\infty}\left(\sum_{j=1}^{N}\frac{6}{N}\left(1+\frac{2j}{N}\right)^2\right)$$

a) Rewrite this expression as a definite integral.

b) Evaluate this integral (using the Fundamental Theorem of Calculus).

7) (8 pts) Cops and robbers.

A person is mugged at the corner of a north-south street and an east-west street, and calls for help. A little while later, at time $t = 0$, the robber is 200m east of the intersection, running east with a speed of 5m/s. (m means "meter" and s means "second") At the same time, a cop is 500m north of the intersection, running south at 5m/s.

a) At what rate is the (straight-line) distance between the cop and the robber changing at $t = 20$ seconds?

b) At what time is the distance between the cop and the robber minimized? (You can restrict your attention to the interval $0 \le t \le 100$ seconds.)

8. (8 points) Compute (with justification) the two limits

a)

$$\lim_{x\to 1}\frac{\sin(\pi x)}{\ln(x)}$$

b)

$$\lim_{x\to\infty}\frac{\sin(x^2)}{x}.$$

9) (8 points) The acceleration of a particle is given by the function $a(t) = 4\cos(t) - 6t + 6$.

a) At $t = 0$, the velocity is $v(0) = 1$. Find the velocity $v(t)$ as a function of time.

b) Let $x(t)$ denote the position at time $t$. Compute $x(2) - x(0)$. (There is more than one way to do this, but there is only one right answer.)