What is a limit?

The general form of a limit statement is

$$\lim_{x \to \text{something}} f(x) = \text{Something else},$$

and means "when x does something, $f(x)$ does something else". There are actually 5 possibilities for "something" and three for "something else", for 15 cases in all, but you should *not* try to memorize all 15 cases separately. You should understand the overall idea of a limit, and then plug that idea into each case.

The possibilities for "something" are:

1. $\lim_{x \to a}$ describes what happens when x is close to, but not equal to, a. So $\lim_{x \to 3}$ involves looking at $x = 3.1, 3.01, 3.001, 2.9, 2.99, 2.999$, and generally considering all values of x that are either slightly above or slightly below 3.

2. $\lim_{x \to a^+}$ describes what happens when x is slighly greater than a. That is, $\lim_{x \to 3^+}$ involves looking at $x = 3.1, 3.01, 3.001$, etc., but not 2.9, 2.99 or 2.999.

3. $\lim_{x \to a^-}$ describes what happens when x is slightly less than a, ignoring what happens when x is slightly greater than a.

   Note that if something happens as $x \to a^+$ and the same thing happens as $x \to a^-$, then that thing happens as $x \to a$. Conversely, if something happens as $x \to a$, then that also happens as $x \to a^+$ and as $x \to a^-$.

4. $\lim_{x \to \infty}$ describes what happens when x grows without bound in the positive direction. The word "infinity" literally means "without end". Imagine taking bigger and bigger values of x, like a hundred, a thousand, a million, a billion, and so on, and seeing what $f(x)$ does. For instance, the statement $\lim_{x \to \infty} f(x) = 7$ means that, whenever x is large and positive, $f(x)$ is close to 7.

5. $\lim_{x \to -\infty}$ is the same thing, only in the negative direction. Look at x being MINUS a million, minus a billion, minus a trillion, etc.

The possibilities for "something else" are:

1. $\lim f(x) = L$ means that $f(x)$ is close to the number L. This is the most common type of limit.

2. $\lim f(x) = \infty$ means that $f(x)$ grows without bound, eventually becoming bigger than any number you can name. Remember that ∞ is not a number! Rather, ∞ is a process of growth that never ends.

3. $\lim f(x) = -\infty$ means that $f(x)$ goes extremely negative and never comes back, eventually becoming less than any number (say, minus a trillion) that you care to name.

Put these together and you can make sense of any limit statement. For instance:

$\lim_{x \to 4^-} \frac{1}{4-x} = \infty$ means that, whenever x is slightly less than 4, $\frac{1}{4-x}$ is gigantic and positive. The graph $y = \frac{1}{4-x}$ will shoot upwards on the left side of the vertical asymptote $x = 4$.

$\lim_{x \to -\infty} 13(e^x + 1) = 13$ means that $f(x) = 13(e^x + 1)$ is close to 13 whenever x is large and negative. So $f(-1,000,000)$ will be close to 13, and $f(-1,000,000,000)$ will be *really* close to 13. The graph $y = f(x)$ will have a horizontal asymptote of $y = 13$ on the left hand side.

$\lim_{x \to \infty} e^x = \infty$ means that, whenever x is large and positive, then e^x will be large and positive (which is true).

I sometimes capitalize the "l" of "lim", writing "Lim" instead of "lim". Lim means the exact same thing as lim.