M408N Second Midterm Exam Solutions, October 27, 2011

1) (48 points, 2 pages) Compute the derivatives of the following functions with respect to x. Except in part (e), you do not need to simplify.

a) $x^2 \ln(x)$

Apply the product rule to get $2x\ln(x) + x = x(1 + 2\ln(x))$

b) $\dfrac{\tan^{-1}(x)}{x^2+1}$

Apply the quotient rule to get

$$\frac{(x^2+1)\frac{1}{x^2+1} - 2x\tan^{-1}(x)}{(x^2+1)^2} = \frac{1 - 2x\tan^{-1}(x)}{(1+x^2)^2}$$

c) $\sin^5(\ln(e^x + 7))$.

Apply the chain rule several times in a row. The derivative is
$5\sin^4(\ln(e^x+7))\frac{d}{dx}(\sin(\ln(e^x+7))$
$= 5\sin^4(\ln(e^x+7))\cos(\ln(e^x+7))\frac{d}{dx}(\ln(e^x+7))$
$= 5\sin^4(\ln(e^x+7))\cos(\ln(e^x+7))e^x/(e^x+7)$.

d) $\dfrac{e^{3x}}{x^2+\cos(5x)}$

Apply the quotient rule and then the chain rule (to get the derivatives of e^{3x} and $\cos(5x)$). The answer is

$$\frac{3e^{3x}(x^2+\cos(5x)) - e^{3x}(2x - 5\sin(5x))}{(x^2+\cos(5x))^2}$$

e) $\sin^{-1}(\cos(x))$, with $0 < x < \pi/2$. Simplify your answer as much as possible!

There are two ways to do this. The first is to notice that $\sin^{-1}(\cos(x)) = \frac{\pi}{2} - x$, whose derivative is -1. The second is to apply the chain rule to get $\frac{1}{\sqrt{1-\cos^2(x)}}(-\sin(x)) = -\sin(x)/\sin(x) = -1$.

f) $x^{(x^2)}$

Use logarithmic differentiation. If $y = x^{x^2}$, then $\ln(y) = x^2\ln(x)$, so $y'/y = (\ln(y))' = (x + 2x\ln(x))$ (as in part (a)), so $y' = y(\ln(y))' = x^{x^2}(x + 2x\ln(x)) = x^{x^2+1}(1 + 2\ln(x))$. Note: THERE IS NO POWER RULE for expressions of the form $f(x)^{g(x)}$! The power rule only applies to expressions like u^n, where n is a *constant*.

2) The curve $x^2 \ln(y) + ye^{x-3} = 1$ goes through the point $P = (3, 1)$. Find the equation of the line that is tangent to the curve at P.

Taking a derivative of the equation with respect to x gives

$$2x \ln(y) + \frac{x^2}{y} y' + ye^{x-3} + e^{x-3} y' = 0.$$

Solving for y' gives

$$y' = \frac{-(2x \ln(y) + ye^{x-3})}{\frac{x^2}{y} + e^{x-3}}.$$

Plugging in $x = 3$ and $y = 1$ gives $y' = -1/10$. Since our tangent line has slope $-1/10$ and goes through $(3, 1)$, its equation is $y - 1 = -(x - 3)/10$, or $y = \frac{-x}{10} + \frac{13}{10}$.

3) Estimate $\sqrt{628}$ and $\sqrt{623}$, each to within .01. (Hint: Use the fact that $\sqrt{625} = 25$.)

Let $f(x) = \sqrt{x}$, so $f'(x) = 1/2\sqrt{x}$. Take $a = 625$, so $f(a) = 25$ and $f'(a) = 1/50$. Our tangent line is then $y - 25 = (x - 625)/50$, or $y = 25 + (x - 625)/50$. At $x = 628$ this gives $y = 25.06$, and at $x = 623$ it gives $y = 24.96$. Not only are these estimates of $\sqrt{628}$ and $\sqrt{623}$ accurate to .01, they are good to within 0.0001, since $\sqrt{628} \approx 25.059928$ and $\sqrt{623} \approx 24.95996$.

4) A F-15 fighter jet is flying 1 km above the ground, and will soon pass directly overhead. It is flying due east at 0.6 km/sec. Where will it be when the distance between you and the plane is decreasing at 0.3 km/sec? That is, how far west of you will the plane be? (Obviously it will still be a kilometer above the ground.) [In case you're interested, here's the physics behind the problem. A jet flying faster than sound generates a sonic boom in your direction when it is approaching you at *exactly* the speed of sound, which is a little over 0.3 km/s. If a jet flies by at Mach 2, it will take a few seconds for the boom to reach you, but the boom will come from exactly the spot that you calculate in this problem.]

Let x be the horizontal distance to the jet, and let r be the diagonal distance, both measured in kilometers. We then have $r^2 = x^2 + 1$, so $2r(dr/dt) = 2x(dx/dt)$. Since $dr/dt = -0.3$ and $dx/dt = -0.6$, we must have $r = 2x$. But then $(2x)^2 = 1 + x^2$, so $3x^2 = 1$, so $x = \sqrt{3}/3$ and the jet is $\sqrt{3}/3$ kilometers to the west. (The direction of the jet is 30 degrees to the west of vertical.)

5) Find all the critical points of the function $f(x) = x^2/(1+x^4)$. Then use these critical points to find the (global) maximum and minimum values of $f(x)$ on the interval $[-10, 10]$.

$f'(x) = [(1+x^4)2x - x^2(4x^3)]/(1+x^4)^2 = 2x(1-x^4)/(1+x^4)^2$. This always exists, and is zero when $x = 0$ or $x = \pm 1$.

Since $f(0) = 0$, $f(1) = f(-1) = 1/2$ and $f(10) = f(-10) = 100/10,001$, the maximum value is $1/2$, and is achieved at $x = \pm 1$, and the minimum value is 0 and is achieved at $x = 0$.

6) (12 points) The position of a particle at time t is given by the function $f(t) = t^3 - 3t$. (a) What are the position, velocity and acceleration of the particle at time $t = -2$?

The velocity function is $f'(t) = 3t^2 - 3$ and the acceleration is $f''(t) = 6t$. Plugging in $t = -2$ we get position $= -2$, velocity $= 9$ and acceleration $= -12$.

b) Indicate all times when the particle is moving forwards (e.g., your answer might be something like "when $t > -7$").

The particle is moving forwards when $|t| > 1$, which is when $f'(t) > 0$. Between $t = -1$ and $t = 1$, the particle is moving backwards.

c) Indicate all times when the particle is accelerating forwards.

The particle is accelerating forwards when $t > 0$, since then $f''(t) = 6t > 0$.