Lie Groups, Problem Set # 7
Due Tuesday, October 30

In this problem set we're going to classify the irreducible root systems. It's a long list of problems, but you can skip the starred exercises that we already did in class.

Recall from lecture (or p119 of the book) that an abstract root system is a subset Φ of a Euclidean space L such that:

1. Φ is a finite set that spans L and does not contain 0. (The elements of Φ are called roots.)

2. For $\alpha \in \Phi$, the only multiples of $\alpha \in \Phi$ are $\pm\alpha$.

3. For $\alpha \in \Phi$, the reflection along α (given by the formula $\beta \mapsto \beta - \frac{2\alpha\cdot\beta}{\alpha\cdot\alpha}\alpha$) maps Φ to itself.

4. For any two $\alpha, \beta \in \Phi$, $2\frac{\alpha\cdot\beta}{\alpha\cdot\alpha}$ is an integer.

1*) Show that the angle between any two roots is either 0, 30, 45, 60, 90, 120, 135, 150 or 180 degrees. Further, show that the two roots have the same length if the angle is 0, 60, 120 or 180 degrees, that the ratio of their lengths is $\sqrt{2}$ if the angle is 45 or 135 degrees, and that the ratio of their lengths is $\sqrt{3}$ if the angle is 30 or 150 degrees.

2) Pick a hyperplane through the origin that does not intersect Φ. We call the roots on one side of this hyperplane positive, and the roots on the other side negative. A positive root is *simple* if it cannot be written as the sum of two other positive roots. Show that the angle between any two simple roots must be at least 90 degrees. [This was mostly done in class. Finish the proof.]

3) For the A_n, B_n, C_n and D_n root lattices, pick the hyperplane $\ell_1 + \epsilon\ell_2 + \epsilon^2\ell_3 + \cdots \epsilon^{n-1}\ell_n = 0$ for sufficiently small ϵ. Show that the positive roots (with the obvious overall sign choice) are exactly the same as those listed in the book. What are the simple roots? [Just list the simple roots – we did the rest in class]

We draw a graph, called a Dynkin diagram, to describe the geometry of the simple roots. Draw one dot for each simple root. Draw single, double, or triple lines connecting two simple roots if the angle between them is 120, 135 or 150 degrees. (Don't draw a line for a 90 degree angle). For the double and triple lines, draw an arrow on the line pointing from the longer root towards the shorter root.

Notation: we call a vertex that is connected to more than two other vertices a *branch point*.

In a root system $\Phi = \Phi_1 \oplus \Phi_2$ is a direct sum of two lower-dimensional root systems, then the simple roots of Φ will be the union of the simple roots of Φ_1 and the simple roots of Φ_2. This will lead to a disconnected Dynkin diagram. If the root system is irreducible, meaning that it isn't a direct sum, then the resulting Dynkin diagram is connected.

4*) From your results in (3), draw the Dynkin diagrams for A_n, B_n, C_n and D_n. Note that B_2 and C_n are given by the same diagram, and that D_3 has the same diagram as A_3. In fact, the Lie algebras are isomorphic. For this reason we usually only consider B_n for $n > 1$, C_n for $n > 2$ and D_n for $n > 3$.

The classification of irreducible root systems proceeds by showing that various sub-diagrams can't occur.

5*) Show that if α and β are connected by a triple line, then α and β are not connected to any other vertices. In other words, the only connected Dynkin diagram with a triple line has only two vertices. This corresponds to the rank-2 group G_2. [Hint: I see two ways to do this, and they're in some sense equivalent. One is to show that there aren't any directions that make at least a 120 degree angle with α and at least a 90 degree angle with β. The other is to assume that there is a root γ with this property and find a linear combination of α, β and γ whose squared length is negative, which is of course a contradiction.]

6*) Show that a connected diagram can have at most one double line. [Hint: assume that you have two double lines and find a linear combination of the roots with negative squared length.]

7) Show that a diagram containing a double line cannot have a branch point.

8*) Show that a diagram with no double lines cannot have any loops.

9*) Show that a diagram with a double line cannot have any loops.

10) Show that, with one exception, you can't have a diagram with a double line that extends in both directions. Specifically, you can't have α connected to β, β connected to γ with a double line (pointing either way), γ connected to δ, and δ connected to ϵ. (If you eliminate the ϵ then the resulting graph with 4 vertices *is* possible, and corresponds to the group F_4.)

11) Draw all the possible diagrams that involve a double or triple line.

We now consider the diagrams where all of the lines are single. These are called the "simply laced" diagrams. The remaining questions are all about simply laced diagrams.

12) Show that any simply laced diagram has at most three ends. In particular, there can be at most one branch point, and it can only be a 3-way intersection.

What remains is to limit the size of the ends.

13) Show that at least one of the branches coming from a branch point has length 1.

14) Show that you cannot have two of the three branches having length 3 or more.

15) Show that if there is a branch point and two of the branches have length 1 and 2, then the third branch has length at most 4.

16) Write down all of the simply laced Dynkin diagrams.

Combining problem 16 with problem 11 gives a complete classification of possible Dynkin diagrams. The ones that aren't A_n or D_n are called E_6, E_7 and E_8, with the subscript indicating how many dots are in the diagram (aka the rank of the group).

The only thing that's missing is the actual Lie theory! The relevant theorems are:

Theorem 1 *Let G be a simple Lie group, let H be a maximal Abelian subgroup consisting of semi-simple elements, and let Φ be the resulting root system. Then the Dynkin diagram for G is connected and does not depend on the choice of H (since all Cartan subgroups are conjugate) and does not depend on the hyperplane used to define positivity.*

Theorem 2 *The same thing goes for simple Lie algebras.*

Theorem 3 *Two simple Lie algebras are isomorphic if and only if they generate the same Dynkin diagram.*

Theorem 4 *All simple Lie algebras are of the form A_n, B_n, C_n, D_n E_6, E_7, E_8, F_4 or G_2. All semi-simple Lie algebras are direct sums of simple Lie algebras. All simply connected semi-simple Lie groups are obtained by exponentiation from semi-simple Lie algebras.*