Lie Groups, Problem Set # 1 Solutions

1) Show that the function $f(t) = \exp(at)[\cos(bt) + i\sin(bt)]$ has $f'(t) = (a+bi)f(t)$ and $f(0) = 1$. In other words, that $\exp((a+bi)t) = f(t)$.

Solution: By the product rule, $f'(t) = a\exp(at)[\cos(bt)+i\sin(bt)]+\exp(at)[-b\sin(bt)+ ib\cos(bt)] = af(t)+ibf(t) = (a+bi)f(t)$. Of course, $f(0) = \exp(0)[\cos(0)+i\sin(0)] = 1$.

2) There are several different norms that can be used for $n \times n$ matrices. These include (a) $\|X\|_1 = \sum_{i,j} |X_{ij}|$, $\|X\|_2^2 = \sum_{i,j} |X_{ij}|^2$, and $\|X\|_{op} = \sup_v |Xv|$, where v ranges over unit vectors in $\mathbb{R}^n$ or $\mathbb{C}^n$, depending on context, and we are using the Euclidean norm for vectors. For each of these norms, show that $\|XY\| \leq \|X\|\|Y\|$, and bound each of these norms by a multiple of the others. (The constants will depend on n).

Solution: (a) $\|XY\| = \sum_{ij} |XY_{ij}| = \sum_{ij} |\sum_k X_{ik}Y_{kj}| \leq \sum_{ijk} |X_{ik}||Y_{kj}| \leq \sum ijkl|X_{ik}||Y_{lj}| = \|X\|_1\|Y\|_1$. (b) By the Schwartz inequality, $|XY_{ij}|^2$ is bounded by the squared norm of the i-th row of X times the squared norm of the j-th column of Y. Summing over i and j gives $\|XY\|_2^2 \leq \|X\|_2^2\|Y\|_2^2$. (c) For any nonzero vector v, $|Yv| \leq \|Y\|_{op}|v|$ and $|X(Yv)| \leq \|X_{op}\||Yv|$, so $|XYv| \leq \|X\|_{op}\|Y\|_{op}|v|$. Restricting to unit vectors and taking a maximum we get that $\|XY\|_{op} \leq \|X\|_{op}\|Y\|_{op}$.

First we compare the $\|\cdot\|_1$ and $\|\cdot\|_2$ norms. Note that $\|X\|_1^2 = \sum_{ijkl} |X_{ij}X_{kl}| \geq \sum_{ij} |X_{ij}|^2 = \|X\|_2^2$, so $\|X\|_2 \leq \|X\|_1$, with equality achieved if and only if at most one matrix element of X is nonzero. Let Z_1 be a matrix whose every element is 1, and suppose that the entries of X are all non-negative real numbers (as this does not affect either $\|\cdot\|_1$ or $\|\cdot\|_2$.) Then $\|X\|_1 = Tr(XZ_1) \leq \|X\|_2\|Z_1\|_2 = n\|X\|_2$, by the Schwartz inequality, since $Tr(XZ_1)$ is the inner product of X and Z_1. The bound $\|X\|_1 \leq n\|X\|_2$ is achieved when all the matrix elements of X have the same magnitude.

Now consider the $\|\cdot\|_{op}$ norm. For any vector v, $|Xv|^2 = v^*X^*Xv$, where * denotes the Hermitian conjugate (transpose of complex conjugate). Since X^*X is a Hermitian matrix, the expression v^*X^*Xv is a weighted average of the eigenvalues of X^*X. This is bounded by the largest eigenvalue, which in turn is bounded by trace of X^*X (since all eigenvalues are non-negative), which is precisely $\|X\|_2^2$. Thus $\|X\|_{op} \leq \|X\|_2$, with equality if (and only if) X has rank 1. This in turn is bounded by $\|X\|_1$, so $\|X\|_{op} \leq \|X\|_1$, with equality if only one element of X is nonzero. Now $|Xe_i| \leq \|X\|_{op}$, so $\|X\|_2^2 = \sum_i |Xe_i|^2 \leq n\|X\|_{op}^2$, so $\|X\|_2 \leq \sqrt{n}\|X\|_{op}$, and hence $\|X\|_1 \leq n^{3/2}\|X\|_{op}$. These estimates are *not* sharp.

3) a) Show that for any matrices X and Y, $\|(X+Y)^n - X^n\| \le (\|X\|+\|Y\|)^n - \|X\|^n$. Here $\|\cdot\|$ denotes any of the norms we discussed in problem 2.

b) Show that, if F is a power series whose radius of convergence is σ, then $F(X)$ is continuous as a function of the matrix X for all $\|X\| < \sigma$.

Solution: a) Expanding $(X+Y)^n - X^n$ gives a sum of $2^n - 1$ terms, each a monomial is X and Y whose norm is bounded by $\|X\|^i \|Y\|^j$, where the monomial contains i powers of X and j powers of Y. Summing these norms gives exactly $(\|X\|+\|Y\|)^n - \|X\|^n$.

b) Let $F(x) = \sum a_n x^n$, and let $\tilde{F}(x) = \sum |a_n| x^n$. F and $\tilde{F}$ have the same radius of convergence, so if $\|X\| < \sigma$ and $\|Y\| < \sigma - \|X\|$, then $\|F(X+Y) - F(X)\| \le \sum \|a_n((X+Y)^n - X^n)\| \le \sum |a_n|((\|X\|+\|Y\|)^n - \|X\|^n)$, which goes to zero as $\|Y\| \to 0$, thanks to the continuity of $\tilde{F}$ as a function of a scalar variable.

4) Prove the Substitution Principle as described on page 13, Remark 2, with the following important modification to part (c). Specifically, assume that $F(z)$ and $G(z)$ are power series with radii of convergence σ and ρ. Let X be a real or complex matrix. Show that
(a) $(F+G)(X) = F(X) + G(X)$ if $\|X\| < \min(\sigma, \rho)$,
(b) $(FG)(X) = F(X)G(X)$ if $\|X\| < \min(\sigma, \rho)$, and
(c) $(F \circ G)(X) = F(G(X))$, if $\|X\| < \rho$, $\|G(X)\| < \sigma$, $G(0) = 0$ *and* $\|X\|$ *is less than the radius of convergence of* $F \circ G$. The left hand sides should be viewed as power series in X with the coefficients given by the appropriate series for scalar functions.

Solution: With the previous problem under our belts, this becomes an exercise in diagonalization. If X is diagonal, then the identities hold one entry at a time. If X is diagonalizable, then after conjugation the same thing holds, since the eigenvalues are all bounded by $\|X\|$. However, the diagonalizable matrices are dense. Since both sides of the identities are continuous, the result extends to all matrices.

You can also get parts (a) and (b) directly. The primary issue is absolute convergence. If a series converges absolutely, then you can rearrange the order of the terms without changing the answer. If it only converges conditionally, then rearranging the terms is not allowed. Note that for any analytic function, the series converges absolutely for arguments less than the radius of convergence, since the terms are bounded by a geometric series. This extends to matrix valued functions, since $\|\sum_{k=n}^{N} c_k X^k\| \le \sum_{k=n}^{N} |c_k| \|X\|^k$, insofar as $\|X^k\| \le \|X\|^k$.

For (a), we have that the series for F and G both converge absolutely, so their sum converges absolutely, so $(f_0 + f_1 X + f_2 X^2 + cdots) + (g_0 + g_1 X + g_2 X^2 + \cdots) = (f_0 + g_0) + (f_1 + g_1)X + (f_2 + g_2)X^2 + \cdots$. The left hand side is $F(X) + G(X)$, while the right hand side is $(F+G)(X)$.

The same reasoning works for (b). Since the sum of the norms of the terms in F and G are each finite, the sum $\sum_{i,j=0}^{\infty} f_i g_j X^{i+j}$ converges absolutely, so we can write it either as $\sum_i \sum_j f_i g_j X^{i+j} = \sum_i f_i X^i \sum_j g_j X^j = F(X)G(X)$, or as $\sum_{k=0}^{\infty} \sum_{i=0}^{k} f_i g_{k-i} X^k$, which is the power series for $(FG)(X)$.

Unfortunately, I couldn't get this approach to work for (c). If any of you succeeded where I failed, more power to you!

[Addendum on Sept 5. The assumptions for part (c) *still* aren't strong enough, and here's a counter-example for scalars. Let $G(x) = e^x - 1$ and let $F(x) = log(x+1)$. Then the power series for $F \circ G$ is just x. Furthermore, $\rho = \infty$, $\sigma = 1$, and $G(2\pi i) = 0 < \sigma$. However, $2\pi i = (F \circ G)(2\pi i) \neq F(G(2\pi i)) = 0$. The correct statement is that the identity $(F \circ G)(x) = F(G(x))$ for scalars has its own radius of convergence η, and that $(F \circ G)(X) = F(G(X))$ applies for matrices when $\|X\| < \eta$. With this assumption, the proof that I gave several paragraphs earlier works fine.]

5) The B-C-H formula allows us to compute Z, where $\exp(Z) = \exp(X)\exp(Y)$, as a sum $Z = X + Y + \sum_{k=2}^{\infty} h_k$, where h_k is a homogeneous polynomial of degree k in X and Y, and where X and Y are assumed to be sufficiently small. The remarkable fact is that h_k is actually a sum of iterated brackets, but in this problem we're just going to think it as a polynomial.
a) Show how to compute h_k iteratively from the previous h's.
b) Use your method to compute h_2 and h_3, and express your answers as (possibly iterated) brackets.

Solution: The order k piece of $\exp(Z)$ is h_k plus various products of lower-order terms. h_k is therefore $\sum_{j=0}^{k} \frac{X^j}{j!} \frac{Y^{k-j}}{(k-j)!}$ minus the products of the lower-order terms. Specifically, the quadratic part of $\exp(Z)$ is $h_2 + (X+Y)^2/2$, so $h_2 = X^2/2 + XY + Y^2/2 - (X+Y)^2/2 = (XY - YX)/2 = [X,Y]/2$, and the cubic part of $\exp(Z)$ is $\frac{(X+Y)^3}{6} + \frac{(X+Y)h_2}{2} + \frac{h_2(X+Y)}{2} + h_3 = \frac{2X^3+5X^2Y+2XYX-YX^2+5XY^2+2YXY-YX^2+2Y^3}{12} + h_3$, so

$$h_3 = \frac{X^2Y - 2XYX + YX^2 + XY^2 - 2YXY + Y^2X}{12} = \frac{[X,[X,Y]] - [Y,[X,Y]]}{12} \tag{1}$$

6) Section 1.2, problems 1, 2

Solution to 1.2.1: If X is nilpotent with $X^k = 0$, then the series for $a = \exp(X)$ terminates after the X^{k-1} term, so there is no issue of convergence. But then $(1-a)$ is a sum of positive powers of X, so $(1-a)^k = 0$, so a is unipotent. Likewise, if a is unipotent with $(1-a)^k = 0$, then the series for $\log(a)$ terminates after $k-1$ terms, and $X = \log(a)$ is a sum of powers of $(1-a)$, so $X^k = 0$. This shows that exp maps the nilpotents to the unipotents and that log maps the unipotents to the nilpotents.

What remains is to show that these are inverse operations. This follows from a modification of the Substitution Principle. We already know that after a suitable rearrangement of terms, the power series of $\log(\exp(x))$ is exactly x. But the series for $\log(\exp(X))$ has only a finite number of nonzero terms, so all rearrangements are OK. Likewise for $\exp(\log(a))$.

Solution to 1.2.2: a) If X is semisimple, then $X = PDP^{-1}$, where D is diagonal and the columns of P are the eigenvectors of X. But then $\exp(X) = P \exp(D) P^{-1}$ is also semisimple, with the same eigenvectors and with eigenvalues that are the exponentials of the eigenvalues of X.

b) If a is invertible and semisimple, then $a = PdP^{-1}$, with d diagonal with all nonzero eigenvalues. But then we can take the logs of all of the diagonal entries of d to get a diagonal matrix D with $\exp(D) = d$. Furthermore we can choose our branch for the log function so that the imaginary part of the entries of D are all in $[0, 2\pi)$. But then $\exp(PDP^{-1}) = PdP^{-1} = a$.

c) First note that this is FALSE if we do not make the assumption about eigenvalues of X not differing by multiples of $2\pi i$. The matrices $X = \begin{pmatrix} 0 & 0 \\ 0 & 2\pi i \end{pmatrix}$ and $X' = \begin{pmatrix} 0 & 1 \\ 0 & 2\pi i \end{pmatrix}$ have $\exp(X) = \exp(X') = I$.

Assuming that no two eigenvalues of X differ by a nonzero multiple of $2\pi i$, I claim that a vector v is an eigenvector of $\exp(X)$ if and only it is an eigenvector of X. The "if" follows from part (a). The "only if" depends on the fact that the exponentials of the eigenvalues of X are all different. If v is a nontrivial linear combination of eigenvectors of X with different eigenvalues, then it is a nontrivial linear combination of eigenvectors of $\exp(x)$ with different eigenvalues, and hence is not an eigenvector of $\exp(X)$.

We now proceed to the proposition. If X and X' are simultaneously diagonalizable, with entries differing by multiples of $2\pi i$, then their exponentials are manifestly the same. Conversely, if $\exp(X) = \exp(X')$, then every eigenvector of X' is an eigenvector of $\exp(X') = \exp(X)$, and hence is an eigenvector of X. Thus, X and X' are simultaneously diagonalizable. For the exponentials of the eigenvalues to agree, the eigenvalues must differ by multiples of $2\pi i$.