Lie Groups Solutions, Problem Set # 10

Section 6.1:

4a: Showing that C is well-defined just means showing that $C((x_1 + x_2) \otimes y) = C(x_1 \otimes y) + C(x_2 \otimes y)$, that $C(x \otimes (y_1 + y_2)) = C(x \otimes y_1) + C(x \otimes y_2)$ and that $C(cx \otimes y) = C(x \otimes cy) = cC(x \otimes y)$, where c is a scalar the x's are elements of V^* and the y's are elements of W. Linearity in x comes from the fact that evaluation at z is a linear map $V^* \to \mathbf{R}$. Linearity in y is obvious (since we're just multiplying by the number $x(z)$. And the way that scalars behave is the linearity of products.

For injectivity, we need the fact that an arbitrary element of $V^* \otimes W$ is a finite sum $\sum_i x_i \otimes y_i$, where the x_i'a are linearly independent, and where the y_i's are linearly independent. (If you have a sum with x's linearly dependent, then you can write one of the x's as a linear combination of the others, expand terms, and get a sum with one fewer term. Repeat this process until you have a minimal sum. Likewise for the y's) But then there exists a z for which $x_1(z) \neq 0$ but $x_i(z) = 0$ for $i > 1$. Then $C(\sum x_i \otimes y_i)(z) = \sum x_i(z) y_i = x_1(z) y_1 \neq 0$, implying that $C(\sum (x_i \otimes y_i)$ is not the zero map.

For surjectivity, we need the additional assumption that V is finite-dimensional. (Not a big deal, since the book only defines tensor products for finite-dimensional spaces.) Let $e_i, \ldots, e_n$ be a basis for V, with dual basis $\phi_1, \ldots, \phi_n$, so that every vector can be expanded as $z = \sum (\phi_i(z)) e_i$. Let $L : V \to W$ be a linear map. Then I claim that $L = C(\sum_i \phi_i \otimes (L(e_i)))$, since $C(\sum_i \phi_i \otimes L(e_i))z = \sum_i \phi_i(z) L(e_i) = L(\sum_i \phi_i(z) e_i) = L(z)$. If V is not finite-dimensional, then the identity map $V \to V$ cannot be written as a (finite) sum of rank-1 maps $C(x \otimes y)$.

This whole construction is best visualized when $V = \mathbf{R}^n$ and $W = \mathbf{R}^m$, in which case $x \in V^*$ is a row vector and y is column vector and $C(x \otimes y)$ is the matrix (outer product) yx.

(b) By definition, $\check{\pi} \otimes \rho(a)(x \otimes y) = (x \circ \pi(a^{-1}) \otimes \rho(a) y)$, so $\check{\pi} \otimes \rho(a)(L) = \rho(a) \circ L \circ \pi(a^{-1})$. In other words, we act by π on the left and ρ on the right.

5: This is similar to problem 4. In fact, showing linearity in x is identical to 4a, showing linearity in y is identical to showing linearity in x, and the properties of scalar multiplication is obvious. For injectivity, we use the same trick as before. An arbitrary element of $V^* \otimes W^*$ can be written as a finite sum $\sum x_i \otimes y_i$ with all the x"s linearly independent and all the y"s linearly independent. Pick u as we picked z before, so that $x_1(u) \neq 0$ but $x_i(u) = 0$ for $i > 1$. Pick v such that $y_1(v) \neq 0$. Then $B(\sum_i x_i \otimes y_1)(u, v) = x_1(u) y_1(v) \neq 0$. For surjectivity, let $e_1, \ldots, e_n$ be a basis for V and let $f_1, \ldots, f_m$ be a basis for W, and let $\{\phi_i\}$ and $\{\psi_j\}$ be the dual bases. Let L be a bilinear map, and let $L_{ij} = L(e_i, f_j)$. Then $L(u, v) = \sum_{i,j} \phi_i(u) \psi_j(v) L_{ij} =$

$B(\sum_{i,j} L_{ij}\phi_i \otimes \psi_j)(u,v)$, so L is in the image of B.

(b) The representation should be $\check{\pi}\otimes\check{\rho}$, not $\check{\pi}\otimes\rho$. Define:

$$(\check{\pi}\otimes\check{\rho})(a)L(u,v) = L(\pi(a^{-1}u, \rho(a^{-1})v)$$

That is, we act by both π and ρ on the right.

The upshot of problems 4 and 5 is that:

$\pi\otimes\rho$ acts on $V\otimes W$ by $(\pi\otimes\rho(a))(v\otimes w) = \pi(a)v\otimes\rho(a)w$

$\check{\pi}\otimes\rho$ acts on $L(V,W) = V^*\otimes W$ by $(\pi\otimes\rho(a))L = \rho(a)\circ L\circ\pi(a^{-1})$

$\check{\pi}\otimes\check{\rho}$ acts on $Bil(V,W) = V^*\otimes W^*$ on the right with respect to both arguments.

Every time you see a V or a W, you act by $\pi(a)$ or $\rho(a)$ on the left, and every time you see a V^* or W^* you act by $\pi(a^{-1})$ or $\rho(a^{-1})$ on the right.

6: The formula for how s acts is badly typeset. It should read

$$s(x_1\otimes\cdots\otimes x_m) = x_{s^{-1}(1)}\otimes\cdots\otimes x_{s^{-1}(m)}$$

(a) Then $sax = (ax)_{s^{-1}(1)}\otimes\cdots\otimes(ax)_{s^{-1}(m)} = a(x_{s^{-1}(1)})\otimes\cdots\otimes a(x_{s^{-1}(m)}) = asx$. If $a\in GL(n,\mathbf{C})$, then a commutes with s, so a commutes with $c = \sum_s \gamma_s s$. Since $acx = cax \in cT^m(\mathbf{C}^n)$, the space $cT^m(\mathbf{C}^n)$ is $GL(n,\mathbf{C})$-invariant.

(b) Again there's a typo, with the correct formula reading

$$(x_1\otimes\cdots x_m, y_1\otimes\cdots\otimes y_m) = (x_1,y_1)\cdots(x_m,y_m)$$

This is linear in x (since each inner product is linear in x_i), is linear in y (ditto), is symmetric. Positivity is slightly trickier. If $x = (x_1\otimes\cdots x_m)$ then it's obvious that $(x,x)\geq 0$. However, a general element of $T^M(\mathbf{C}^n)$ is NOT a product $x_1\otimes\cdots\otimes x_m$. It's a linear combination of such terms. If $x = \sum_i(x_1^i\otimes\cdots\otimes x_m^i)$, then it's not so obvious that $(x,x)\geq 0$. Still, it's true. To see this, consider products of the form $e_{i_1}\otimes\cdots\otimes e_{i_m}$, where each e_{i_j} is a standard basis vector. These products are orthonormal, so the inner product is positive on the span of these products. But that's all of $T^m(\mathbf{C}_n)$, since

$$x_1\otimes\cdots\otimes x_n = \sum x_1^{i_1}x_2^{i_2}\cdots x_m^{i_m}e_{i_1}\otimes\cdots\otimes e_{i_m},$$

where we have expanded each vector x_k as $\sum_{i_k} x_k^{i_k}e_{i_k}$.

Now, for x and y being simple products, $(ax,y) = \prod(ax_i,y_i) = \prod(x_i,a^*y_i) = (x,a^*y)$. This extends by linearity to all of $T^m(\mathbf{C}^n)$, not just the simple products.

In particular, if a is unitary then $(ax, ay) = (x, y)$, so $U(n)$ acts by unitary transformations of $T^m(\mathbf{C}^n)$. Now suppose that V is a linear subspace of $T^m(\mathbf{C}^n)$ that is preserved by $GL(n, \mathbf{C})$. Then V is preserved by $U(n)$, so $V^\perp$ is preserved by $U(n)$. But this means that V and $V^\perp$ are preserved by the Lie algebra $u(n)$, and hence also by $iu(n)$, since we're dealing with a linear group. But then V and $V^\perp$ are preserved by $\exp(u(n) \oplus iu(n) \subset GL(n, \mathbf{C})$, and so are preserved by all of $GL(n, \mathbf{C})$ (since an arbitrary element is a product of exponentials). This implies that a reducible representation is decomposable. Decomposing as many times as necessary (a finite number, since $T^m(\mathbf{C}^n)$ is finite-dimensional), we get a completely reduced representation.

7a) If $\tilde{s} \in S$, then $\tilde{s}\mathrm{sym} = \frac{1}{m!}\sum_s \tilde{s}s = \frac{1}{m!}\sum_{s'} s' = \mathrm{sym}$, where $s' = \tilde{s}s$. Hence $\tilde{s}$ is the identity on $\mathrm{sym}(T^m(\mathbf{C}^n))$, and $sym = \frac{1}{m!}\sum \tilde{s}$ is also the identity. Since $(sym)^2 = sym$, sym is a projection onto its image.

(b) Since the tensor products $e_{i_1} \otimes \cdots \otimes e_{i_m}$ is a basis for $T^m(\mathbf{C}^n)$, sym of this set spans $S^m(\mathbf{C}^n)$. But $sym(e_1 \cdots e_m) = sym(e_1^{m_1} e_2^{m_2} \cdots e_n^{m_n})$, where we have just rearranged the order of the terms to put all of the e_1's first, then all of the e_2's, etc. Thus the sym-monomials are a spanning set. It's not hard to see that they're orthogonal, using the inner product from (6), so they're linearly independent.

(c) We just have to count the number of sym-monomials. This is the same as the number of ways to put m items into n slots, repetitions allowed, with order irrelevent. You may remember this problem from a probability class, but in case you don't here's the derivation. That's the number of ways to come up with a non-decreasing sequence of m integers between 1 and n whose sum is m. By adding 0 to the first term, 1 to the second, etc, this turns into the number of ways to write a strictly increasing sequence of m integers between 1 and $n+m-1$. But that's the same as picking m items from a set of $n+m-1$ elements. So the answer is $\binom{m+n-1}{m}$.

(d) Without loss of generality we can assume that a is diagonal, since the trace is invariant under conjugation. But for a diagonal matrix, the sym-monomials are eigenvectors with eigenvalue $\epsilon_1^{m_1} \cdots \epsilon_n^{m_n}$ (not $\epsilon_n^{m_1}$ – yet another typo). Summing over our basis gives the result.

Again assume that a is diagonal. Then $det(1-za)^{-1} = \prod(1-z\epsilon_i)^{-1} = \prod_i(\sum_{m_i} \epsilon_i^{m_i} z^{m_i})$. The coefficient of z^m is then $\psi_m(a)$.

11. (a) There are several ways to do this. One is by brute force. Define $y = a^{-1}x$, so $x = ay$. We need to prove that $\sum_i \partial^2/\partial x_i^2 = \sum_j \partial^2/\partial y_j^2$. Since $\frac{\partial}{\partial y_j} = \sum_i \frac{\partial x_i}{\partial y_j}\frac{\partial}{\partial x_i} = \sum_i a_{ij}\frac{\partial}{\partial x_i}$. But then

$$\sum_j \frac{\partial^2}{\partial y_j^2} = \sum_{i,j,k} a_{ij}a_{kj}\frac{\partial^2}{\partial x_i \partial x_k} = \sum_{i,k}(aa^T)_{ik}\frac{\partial^2}{\partial x_i \partial x_k} = \sum_i \frac{\partial^2}{\partial x_i^2}$$

(b) The only real issue is $SO(n)$-invariance, which is because the measure on ν on S^{n-1} is $SO(n)$-invariant.

(c) Since $\Delta(x_1 + ix_2)^m = 0$ (by direct computation, since $1^2 + i^2 = 0$), the kernel of Δ is an invariant subspace. But it isn't all of $R^m(\mathbf{R}^n)$, since $\Delta(x_1^m) \neq 0$. Thus the representation is reducible.

Section 6.2

1. By the Peter-Weyl theorem, we can write $f = \sum_{i,j,\lambda} f_{ij}^\lambda \pi_{ij}^\lambda$. If this sum involves only a finite number of irreducible representations λ, then the left-translates of $L(a)f(x) = f(a^{-1}x) = \sum_{\lambda,i,j,k} f_{ij}^\lambda \pi_{ik}^\lambda(a^{-1})\pi_{kj}^\lambda(x)$ live in a finite-dimensional space, as do the right-translates $R(a)f(x) = f(xa) = \sum f_{ij}^\lambda \pi_{ik}^\lambda(x)\pi_{kj}^\lambda(a)$. In fact they live in the same finite-dimensional space, spanned by all of matrix elements of the representations that appear in the sum. In that case, we can pick up a single representation, namely the direct sum of these irreps, with each irrep of dimension n counted n^2 times. If we let a component of x be e_i and the component of y be $f_{ij}^\lambda e_j^*$, then the corresponding contribution to $< \pi(a)x, y >$ will be $f_{ij}^\lambda \pi_{ij}^\lambda(a)$. Finally, since all functions of G are approximated by polynomial functions, and polynomial functions fall into finite-dimensional irreducible representations, being a polynomial is equivalent to having components in only a finite number of representations.

On the other hand, if our sum involves infinitely many terms, it involves infinitely many representations. The left translates cannot span a finite dimensional space, since that space could be written as a finite number of irreps, and its matrix elements would come from a finite sum of π_{ij}^λ's. Same for right-translates. Since (c) and (d) manifestly imply (a) and (b), the negation of (a) and (b) imply the negation of (c) and (d). Thus (a), (b), (c), and (d) are equivalent.

2.

$$\begin{aligned}
\left(\sum_a f(a)a\right)\left(\sum_a g(a)a\right) &= \left(\sum_a f(a)a\right)\left(\sum_b g(b)b\right) \\
&= \sum_{a,b} f(a)g(b)ab \\
&= \sum_{b,c} f(cb^{-1})g(b)c \text{ where } c = ab \\
&= \sum_c (f*g)(c)c = \sum_a (f*g)(a)a
\end{aligned}$$

$$(f*g)*h(a) = \int_G (f*g)(ab^{-1})h(b)db$$

$$
\begin{aligned}
&= \int_G \int_G f(ab^{-1}c^{-1})g(c)h(b)dbdc \\
&= \int_G \int_G f(ad^{-1})g(db^{-1})h(b)dbdd \text{ where } d = cb \\
&= \int_G f(ad^{-1})(g * h)(d)dd \\
&= f * (g * h)(a)
\end{aligned}
$$

$$
\begin{aligned}
\pi(f * g) &= \int_G (f * g)(a)\pi(a) \\
&= \int_G \int_G f(ab^{-1})g(b)\pi(a)dadb \\
&= \int_G \int_G f(c)g(b)\pi(cb)dcdb \text{ where } c = ab^{-1} \\
&= \int_G \int_G f(c)g(b)\pi(c)\pi(b) = \pi(f)\pi(g)
\end{aligned}
$$

(d) First we note that $\rho(\pi_{ij}^{\lambda}) = 0$ if and only if $\lambda \not\simeq \rho$. This follows from the orthogonality relations. So if $f, g \in M(G)$, then $\pi(f)$ and $\pi(g)$ are zero for all but a finite number of representations. But then $\pi(f * g) = \pi(f)\pi(g)$ is zero for all but finitely many representations. But then $f * g$ only has finitely many components and is in $M(G)$.