Lie Groups Solutions, (Optional) Problem Set # 11

Section 6.3:

Problem 6.3.1: (a) Since the irreducible characters span the space of class functions, the number of such characters is the dimension of this space. But a class function is determined by its value on each conjugacy class, so the dimension of the space of class functions is the number of classes.

(b) If ρ is the left (or right) -regular representation, the trace of $\xi^\rho(a)$ is the number of elements of the group fixed by left (or right) multiplication by a (or a^{-1}). This is $|G|$ (NOT $1/|G|$) if $a = 1$ and 0 otherwise, since $ax = x$ if and only if $a = 1$.

(c) A basis for $\mathbf{C}^X$ is the indicator functions of each point x. Since the action of the group permutes these functions, the trace of $\pi(a)$ is the number of functions that are left fixed, which is the set of x for which $a^{-1}x = x$, which is the set of x for which $x = ax$, which is $|X^a|$.

6.3.3: Since G is compact, we can assume that $\pi(a)$ is unitary, hence diagonalizable with eigenvalues $\lambda_1, \ldots, \lambda_n$. If x and y are eigenvectors of $\pi(a)$ with eigenvalues λ_i and λ_j, then $x \otimes y + y \otimes x$ is an eigenvector of $Sym^2\pi(a)$ with eigenvalue $\lambda_i\lambda_j$. Note that we can assume $i \leq j$, but equality IS allowed. Then

$$Sym^2\chi(a) = \sum_{i \leq j} \lambda_i\lambda_j,$$

which counts each distinct i, j once and each repeated index once. Meanwhile, $\chi(a)^2 = \sum_i \lambda_i \sum_j \lambda_j = \sum_{i,j} \lambda_i\lambda_j$. This sum counts each pair of distinct i, j twice and each repeated index once. Adding $\chi(a^2) = \sum_i \lambda_i^2$ and dividing by two gives $Sym^2\chi(a) = \frac{1}{2}(\chi(a)^2 + \chi(a^2))$.

The eigenvectors of $Alt^2\pi(a)$ are $x \otimes y - y \otimes x$, with eigenvalues $\lambda_i\lambda_j$ with i strictly less than j. The sum of these is $\frac{1}{2}(\chi(a)^2 - \chi(a^2))$

(b) Since $Sym^2\chi(a) + Alt^2\chi(a) = \xi(a)^2$, which is the trace of $\pi(a) \otimes \pi(a)$, we in particular have $Sym^2\chi(1) + Alt^2\chi(1) = n^2$, where n is the dimension of V. But $\chi^\lambda(1)$ gives the dimension of a representation, so the Sym^2 and Alt^2 spaces have total dimension n^2. Since their intersection is trivial, their sum is all of $V \otimes V$. Of course, this is easy to see directly, since $x \otimes y = \frac{1}{2}\left((x \otimes y + y \otimes x) + (x \otimes y - y \otimes x)\right)$.

Section 6.4

Problem 6.4.1: (a) The test for whether a representation is irreducible is whether $\frac{1}{|G|}\int_G |\chi(a)|^2 da = 1$. But $\chi^\lambda(a) = \xi_\kappa(a)/\Delta(a)$, so $\chi^\lambda(e^{i\theta}a) = e^{i\sum \ell_i\theta}\chi^\lambda(a)$, so the average value of $|\chi(a)|^2$ over $U(n)$ is the same as the average value over $SU(n)$. Thus χ^λ, restricted to $SU(n)$, is an irreducible character.

(b) Suppose we have an irreducible representation of $SU(n)$ that does not extend to a representation of $U(n)$, and let χ be its character. In $L^2(SU(n))$, χ is orthogonal to all of the other irreducible characters of $SU(n)$, and in particular to the restrictions of the irreducible characters of $U(n)$. We will extend χ to a class function on $U(n)$ that is orthogonal to all of these characters of $U(n)$. That will be a contradiction, since the characters of $U(n)$ span the class functions on $U(n)$.

Every matrix $a \in U(n)$ whose determinant is close to 1 (say, with positive real part) can be uniquely written as a product $e^{i\theta}b$, where $b \in SU(n)$ and $|\theta| < \pi/2n$. Let g be a bump function with support in $(-\pi/2n, \pi/2n)$, and let $f(a) = g(\theta)\chi(b)$ for a's whose determinants have positive real part, and let $f(a) = 0$ if $\det(a)$ has negative or zero real part. Then f is our desired class function that is orthogonal to all of the irreducible characters of $U(n)$.

(c) Acting on a diagonal matrix t, $ei^\lambda(t) = \prod \epsilon_i^{l_i} = \prod_{i=1}^{n-1} \epsilon_i^{l_i - l_n}$ on $SU(n)$, since $\epsilon_n = (\prod_{i=1}^{n-1} \epsilon_i)^{-1}$. If $\mu - \lambda$ is a multiple of $(1, 1, 1, \ldots)$, then $e^\lambda = e^\mu$ on $SU(n)$, so $\xi^\lambda = \frac{1}{\Delta} \pm e^{s(\lambda+\rho)} = \frac{1}{\Delta} \pm e^{s(\mu+\rho)} = \xi^\mu$, and the characters are the same. Conversely, if μ and λ are strictly dominant and don't differ by a multiple of $(1, 1, \ldots, 1)$, then they take on different values on the diagonal torus (one has a component e^λ and the other doesn't), and cannot be equal.

Problem 6.4.2: (a) $\chi_\ell = \xi_{\ell+1}/\Delta = (\epsilon^{\ell+1} - \epsilon^{-\ell-1})/(\epsilon - \epsilon^{-1}) = \epsilon^\ell + \epsilon^{\ell-2} + \cdots + \epsilon^{-\ell}$.

(b) Suppose that $m \leq \ell$ Since the character of $V_l \otimes V_m$ is $\chi_l \chi_m$, so $\xi^\lambda = \xi_l \chi_m = (\epsilon^{\ell+1} - \epsilon^{-(\ell+1)})(\epsilon^m + \epsilon^{m-2} + \cdots + \epsilon^{-m})$ Multiplying this out, we get the sum of the ξ_k's for k ranging from $\ell + m$ to $\ell - m$ in steps of 2. This means that $V_\ell \otimes V_m$ must be the direct sum of those V_k's.

Section 6.5

Problem 6.5.1: We started to work this problem in class. We have $[X_+, X_-] = H$, $[H, X_+] = 2X_+$ and $[H, X_-] = -2X_-$. Pick an irreducible representation ρ and let $\tilde{H} = \rho(H)$, $\tilde{X}_+ = \rho(X_+)$ and $\tilde{X}_- = \rho(X_-)$, so $[\tilde{X}_+, \tilde{X}_-] = \tilde{H}$, $[\tilde{H}, \tilde{X}_+] = 2\tilde{X}_+$ and $[\tilde{H}, \tilde{X}_-] = -2\tilde{X}_-$. If v is an eigenvector of $\tilde{H}$ with eigenvalue λ, then

$$\tilde{H}\tilde{X}_\pm v = (\tilde{X}_\pm \tilde{H} + [\tilde{H}, \tilde{X}_\pm])v = (\tilde{X}_\pm)(\tilde{H} \pm 2)v = (\lambda \pm 2)\tilde{X}_\pm v,$$

so either $\tilde{X}_\pm v$ is an eigenvector with eigenvalue $\lambda \pm 2$ or $\tilde{X}_\pm v = 0$. We call $\tilde{X}_\pm$ *raising and lowering operators*, or *ladder operators*.

Let ℓ be the largest eigenvalue of $\tilde{H}$, and let v_ℓ be the corresponding eigenvector. Let $v_{\ell-2} = \tilde{X}_- v_\ell$, and define recursively $v_{k-2} = \tilde{X}_- v_k$. Since our representation is finite-dimensional, the sequence $v_\ell, v_{\ell-2}, \ldots$ must eventually terminate in a vector $v_{\ell'}$ with $\tilde{X}_- v_{\ell'} = 0$.

Now consider the matrix $C = \tilde{H}^2 + 2\tilde{X}_-\tilde{X}_+ + 2\tilde{X}_+\tilde{X}_- = \tilde{H}^2 + 2\tilde{H} + 4\tilde{X}_-\tilde{X}_+ = \tilde{H}^2 -$

$2\tilde{H}+4\tilde{X}_+\tilde{X}_-$. (In quantum mechanics, $\tilde{H}$ is called $2J_3$, $\tilde{X}_\pm$ is called $J_\pm$, and $C = 4J^2$. Also, the quantum number j is $\ell/2$.) You can check that $[\tilde{X}_\pm, C] = [\tilde{H}, C] = 0$. This makes C a **g** map. Since the representation ρ is irreducible, Schurr's Lemma says that C is multiplication by a constant. Since $Cv_\ell = (\ell^2 + 2\ell)v$ (since $\tilde{X}_+ v = 0$), this constant must be $\ell^2 + 2\ell$. But $Cv_{\ell'} = (\ell'^2 - 2\ell')v_{\ell'}$, since $\tilde{X}_- v_{\ell'} = 0$. This implies that $\ell' = -\ell$, so our vectors are $v_\ell, v_{\ell-2}, \ldots, v_{-\ell}$.

Since we have $\tilde{H}v_k = kv_k$ and $\tilde{X}_- v_k = v_{k-2}$, we just need to compute $\tilde{X}_+ v_k$ to complete our representation. But $\tilde{X}_+ v_k = \tilde{X}_+\tilde{X}_- v_{k+2} = \frac{1}{4}(C - \tilde{H}^2 + 2\tilde{H})v_{k+2} = \frac{1}{4}(\ell(\ell+2) - k(k+2))v_{k+2}$.

This is not QUITE the same as example 5, but it's close. The only difference is how we normalize the eigenvectors v_k.

Problem 6.5.2. (a) Let $Y \in SO(3)$, and let $a(t) = \exp(Yt)$. The velocity of $a(t)x$ at $t = 0$ is Yx, so the derivative of $f(a(t)x)$ is $\sum_i (Yx)_i \partial_i f = \sum_{i,j} Y_{ij} x_j \partial_j f$. Since Y can be an arbitrary anti-symmetric matrix, this is zero for all Y and all x if and only if $x_i \partial_j f = x_j \partial_i f$ for all pairs (i, j) and for all points x. Since exponential map is onto, having $f(ax) = f(x)$ for all a is equivalent to $f(\exp(Yt)x) = f(x)$ for all Y and all t and all x, which is equivalent to the derivative with respect to t being zero.

(b) The solution as written to part (a) never uses the fact that we are working specifically with $SO(3)$, and applies equally well to $SO(n)$.

Problem 6.5.3: (a) $\pi^\pm(iX) = \frac{1}{2}(\pi(iX) \pm i\pi(X)) = \pm i\frac{1}{2}(\pi(X) \mp i\pi(iX)) = \pm i\frac{1}{2}(\pi(X) \pm i\pi(-iX)) = \pm i\pi^\pm(X)$.

(b) This doesn't seem to make any sense as written. Since (c) depends on (b) and it seems we're supposed to use (b) and (c) to get (d), I'm going to skip the rest of the problem.

Problem 6.5.4. If you have a real representation ρ on **g**, define $\tilde{\rho}(X + iY)$ to be $\rho(X) + i\rho(Y)$ for $X + iY \in \mathbf{g} \oplus i\mathbf{g}$. This is manifestly homolorphic. Conversely, if we have a holomorphic representation $\tilde{\rho}$ of $\mathbf{g} \oplus i\mathbf{g}$, we can restrict it to **g** to get a representation ρ of **g**. It's easy to see that these extensions and restrictions are inverse maps, since (for the extension) $\tilde{\rho}(X + i0) = \rho(X)$ and (for the restriction) $\rho(X) + i\rho(Y) = \tilde{\rho}(X) + i\tilde{\rho}(Y) = \tilde{\rho}(X + iY)$. Equivalence is just conjugation by a fixed matrix, and is preserved by this operation. Irreducibility is easy *as long as we are talking about complex subspaces of* V. A complex subset is preserved by all X if and only if it is preserved by all iY, if and only if it is preserved by all $X + iY$.

Problem 6.5.5: An arbitrary element of $\mathbf{g_C}$ can be written as $\alpha = X + jY$ where $X, Y \in \mathbf{g}$. In other words, we're reserving the letter i for complex multiplication within bg, and define $j(X, Y) = (-Y, X)$ in $bg \oplus bg$, so that $X + jY$ is shorthand for (X, Y). Define $\alpha^\pm = \frac{1}{2}\alpha \mp ji\alpha$. These are the $\pm i$ eigenspaces of the operator j. Each eigenspace is naturally identified with $\mathbf{g}$ via the map $X \to (X + j0)^\pm$. In other words, the complexification of a complex vector space is the direct sum of two copies of that space, only with j acting by i on the first copy and $-i$ on the second.

So far we've only used the fact that $\mathbf{g}$ is a complex vector space. Now we have to check the the algebra operations are satisfied, namely that $[\alpha^\pm, \beta^\pm] = [\alpha, \beta]^\pm$ and that $[\alpha^\pm, \beta^\mp] = 0$. Note that we have $[X_1 + jY_1, X_2 + jY_2] = [X_1, X_2] - [Y_1, Y_2] + j[X_1, Y_2] + j[Y_1, X_2]$.

$$[\alpha^\pm, \beta^\mp] = -[j\alpha^\pm, j\beta^\mp] = -[\pm i\alpha^\pm, \mp i\beta^\mp] = -[\alpha^\pm, \beta^\mp]$$

so $[\alpha^\pm, \beta^\pm] = 0$. We also compute, for $\alpha, \beta \in \mathbf{g}$:

$$[\alpha^+, \beta^+] = \frac{1}{4}[\alpha - ji\alpha, \beta - ji\beta] = \frac{1}{4}(2[\alpha, \beta] - 2ji[\alpha, \beta]) = [\alpha, \beta]^+$$

The calculation for $[\alpha^-, \beta^-]$ is similar.

By the way, here's the simplest example of the complexification of a complex vector space. Start with a 1 complex vector space, namely $\mathbf{C}^1$ with coordinate z. Think of it as $\mathbf{R}^2$ with coordinates x and y, where $z = x + iy$. The cotangent space has basis dx and dy. Now complexify this, by taking complex linear combinations of dx and dy. That's the same thing as taking arbitrary linear combinations of $dz = dx + idy$ and $d\bar{z} = dx - idy$.

Problem 6.5.6: (a) In $SO(3)$, $\exp(i\pi H) = 1$, so the eigenvalues of H must be even integers. These are the representations with ℓ even. (In physics language, this means that $j = \ell/2$ has to be an integer rather than a half-integer.)

(b) $\exp(ad/dx)f(x) = \sum \frac{a^n}{n!} f^{(n)}(x)$. This equals $f(x + a)$ if the function f is analytic, so the exponential of d/dx must be a translation. However, the interval $(0, 1)$ is bounded – translate by more than 1 and you're off the interval! In particular, the translate of a bump function is zero. (Note: the bump function isn't analytic, but it is smooth.) Thus is doesn't make sense to exponentiate d/dx on $C^\infty(0, 1)$.

On $C^\infty(\mathbf{R})$, however, we can define the translation operator $T_t f(x) = f(x + t)$. Since $dT_t/dt = T(d/dx) = (d/dx)T$, we can reasonably say that $T_t = \exp(td/dx)$. However, this exponentiation is NOT given by a power series, since Taylor series only works for analytic functions, not for arbitrary smooth functions.

(c) Vermer modules are a bridge too far.