Lie Groups Homework 3 Solutions

Problems from Section 2.3

2.3.4 Remember that we always take $k = \infty$ (although the argument works just as well for finite k). Suppose we are working with local coordinates where $a = a_0 \exp(X)$. If f is a smooth function of all matrices in a neighborhood of a_0, then $f(a) = f(a_0 \exp(X))$ is a composition of smooth functions, and so is smooth. For the converse, consider the group of matrices of the form $\begin{pmatrix} \exp(it) & 0 \\ 0 & \exp(\sqrt{2}it) \end{pmatrix}$. The function "$t$" is smooth as a function on the group, but cannot be extended to a continuous function on $GL(2, \mathbb{C})$.

2.3.5. THIS IS WRONG AS STATED! I apologize for not giving you a warning in time. The intervals given only give half of $SU(2)$, mapping to $SO(3)$ once.

Let $\tilde{a}$ be the image of a under the homomorphism $SU(2) \to (SO(3)$. It is easy to check that this homomorphism sends the $a_{2,3}$ functions of this problem to the $a_{2,3}$ functions of Example 4. Since $\tilde{a}$ can be written as $a_3(\theta)a_2(\phi)a_3(\psi)$ (using the $SO(3)$ functions), one of the preimages of $\tilde{a}$ can be similarly written using the $SU(2)$ functions. That is, $\pm a = a_3(\theta)a_2(\phi)a_3(\psi)$.

2.3.9: This is part of Theorem 1 of section 2.6, and a proof can be found on page 78. We also just did it in class. For completeness, however, I'll reprise the argument here.

Let $g(t) = f(\exp(tX))$. Then $g(0) = 1$ and $g'(t) = d(f(\exp((t+s)X)/dx)|_{s=0} = d(\exp(tX)\exp(sX))/ds|_{s=0} = g(t)\phi(X)$. But the solution to this differential equation is $\exp(t\phi(X))$ satisfies, so $f(\exp(tX)) = \exp(t\phi(X))$. Finally, set $t = 1$.

2.3.10: (a) This was essentially done in the proof of Theorem 3. We constructed the analytic map $f(X, Y) = \exp(X)(1 + Y)$ from M to M, and noted that by the inverse function theorem it had an analytic local inverse near 1. Since the leaves of G were $V =$ constant (with V denoting the function in the proof, NOT the open ball in $\mathbb{R}^N$), and $G \cup U$ is C^1-path-connected, this means that every point in $G \cup U$ has $V = 0$, and hence maps to a ball around zero in $\mathbb{R}^m \times 0 \subset \mathbb{R}^N$.

(b) Part (a) showed that $1 \in G$ has a neighborhood in G which is the restriction of an open set (in M) to G. Multiplying on the left (or right) by a then gives us a neighborhood of $a \in G$ with the same property. This shows that the intrinsic topology of G is the same as the topology that G inherits from M.

(c) First work locally, then glue. Locally, if we have a C^k function on G, then it gives a C^k function on $\mathbb{R}^m$, which, when multiplied by a smooth function of the remaining $N-m$ coordinates, gives a C^k function on $\mathbb{R}^N$, hence on a neighborhood of a in M. Now glue these local functions together using a smooth partition-of-unity of M. This shows that any C^k function on G can be extended to a C^k function on M. The fact that the restriction of a C^k function on M to G is C^k is trivial.

Section 2.5 Problems

2.5.2: As usual, we convert statement about G into statements about $\mathbf{g}$ by differentiation, and statements about $\mathbf{g}$ into statements about G by exponentiation.

Suppose that F is S-stable. Then for any path $a(t)$ in G with $a(0) = 1$ and $a'(0) = X$, and any vector $v \in F$, $a(t)v \in F$. Taking a derivative w.r.t. t at $t = 0$ gives $Xv \in F$, so F is $\mathbf{g}$-stable. Conversely, if F is $\mathbf{g}$-stable, then X maps F to F, so $\exp(X)$ maps F to F so $\Gamma(\mathbf{g})$ maps F to F. Since G is connected, $\Gamma(\mathbf{g}) = G$.

2.5.3: We follow the suggestion in the book. Since the space of coboundaries is a vector space (isomorphic to $\mathbf{g}^*$), and since $\omega^{a(1)} - \omega^{a(0)} = \int_0^1 \frac{d}{dt}\omega^{a(t)}dt$, it is enough to show that $\frac{d}{dt}\omega^{a(t)}$ is a coboundary. However, if $a'(t) = a(t)Z$, then $\frac{d}{dt}Ad(a) = Ad(a) \circ ad(Z)$. Letting $\tilde{X} = Ad(a)X$, $\tilde{Y} = Ad(a)Y$ and $\tilde{Z} = Ad(a)Z$, we compute

$$\begin{aligned}
\frac{d}{dt}\omega^a(X,Y) &= \omega(Ad(a)ad(Z)X, Ad(a)Y) + \omega(Ad(a)X, Ad(a)\circ ad(Z)Y) \\
&= \omega([\tilde{Z},\tilde{X}],\tilde{Y}) + \omega(\tilde{X},[\tilde{Z},\tilde{Y}]) \\
&= \omega([\tilde{Z},\tilde{X}],\tilde{Y}) + \omega([\tilde{Y},\tilde{Z}],\tilde{X}) \\
&= \omega(\tilde{Z},[\tilde{X},\tilde{Y}]) = \omega^a(Z,[X,Y])
\end{aligned}$$

which is a linear function of $[X,Y]$, and hence is a coboundary.

2.5.5: Going through the list of groups mentioned in 2.1: All of the groups mentioned prior to $SL(2,C)$ are odd-dimensiona as real spaces, so their Lie Algebras can't possibly be invariant under multiplication by i. $SL(2,C)$ is complex, as its Lie Algebra is the space of traceless matrices, which is a complex vector space. The only other comlpex spaces are the additive groups C^n and $GL(E)$, where E is a complex vector space.

2.5.6: Suppose that G is a compact linear group, and let $X \in \mathbf{g}$. Since the matrices $\exp(tX)$ must be bounded, the eigenvalues of X must be pure imaginary. If G is complex, then the eigenvalues of iX must likewise be pure imaginary. In other words, all of the eigenvalues of X must be zero. But then X is nilpotent, so $\exp(tX)$ is a polynomial in t. This polynomial is bounded if and only if all of the coefficients are zero, which means that X must be the zero matrix. Since $\mathbf{g}$ is trivial, $G = \Gamma(\mathbf{g})$ must be the trivial group. (Note that we are using the fact that G is connected, hence that $G = \Gamma(\mathbf{g})$.)

2.5.11: (a) The fact that Z is self-adjoint is essential, because that means that Z is diagonalizable with real eigenvalues. Working in a basis where Z is a diagonal matrix (say with elements λ_i on the diagonal), then $[Z, X]$ is a matrix whose ij element is $(\lambda_j - \lambda_i)X_{ij}$. That is, $ad(Z)$ acting on the space of ALL matrices is diagonalizable with real eigenvalues, so there do not exist any matrices X for which $(ad(Z) - \lambda)^p X = 0$ but $(ad(Z) - \lambda)X \neq 0$. Hence $ad(Z)$ is still diagonalizable when retricted to $\mathbf{g}$, which is the same thing as saying $\mathbf{g} = \oplus_\lambda \mathbf{g}_\lambda$.

(b): If $X \in \mathbf{g}_\lambda$ and $Y \in \mathbf{g}_\mu$, then $[Z, [X, Y]] = [[Z, X], Y] + [X, [Z, Y]] = [\lambda X, Y] + [X, \mu Y] = (\lambda + \mu)[X, Y]$, so $[X, Y] \in \mathbf{g}_{\lambda + \ mu}$.

(c) If $[Z, X] = \lambda X$, then $[Z, X^*] = [Z^*, X^*] = [X, Z]^* = (-\lambda X)^* = -\lambda^* X^* = -\lambda X^*$, since $\lambda = \lambda_i - \lambda_j$ is real.

(d) $\mathbf{k}$ is a sub-algebra since the commutator of two anti-Hermitian matrices is anti-Hermitian. ($[X, Y]^* = [Y^*, X^*] = [-Y, -X] = [Y, X] = -[X, Y]$) $\mathbf{q}$ is a sub-algebra by (b). Now suppose that $Y \in \mathbf{g}$. We write $Y = Y_+ + Y_-$, where Y_+ is the projection of Y onto the non-negative eigenspaces of $ad(Z)$, and Y_- is the projection onto the negative eigenspaces. Then Y_1 and Y_2^* are both in $\mathbf{q}$, and we can write $Y = (Y_1 + Y_2^*) + (Y_2 - Y_2^*) \in \mathbf{q} + \mathbf{k}$. Note that the sum $\mathbf{g} = \mathbf{k} + \mathbf{q}$ is not necessarily a DIRECT sum. If $Y \in \mathbf{g}_0$, then $Y - Y^*$ is in both k and q.

(e) K is the intersection of G with $U(n)$ or $O(n)$, so the Lie Algebra of K is the intersection of $\mathbf{g}$ with the anti-Hermitian matrices. In other words, $L(K) = \mathbf{k}$. By Proposition 10, $L(Q)$ is the normalizer of $\mathbf{q}$. Note that $Z \in \mathbf{g}_0 \subset \mathbf{q}$. If X is in the normalizer of $\mathbf{q}$, then $[X, Z] = -ad(Z)(X)$ must be in q. But this is only possible if X is already in $\mathbf{q}$. Thus the normalizer of $\mathbf{q}$ is contained in $\mathbf{q}$. However, $\mathbf{q}$ is a sub-algebra, so it is contained in its normalizer, so $n_{\mathbf{g}}(\mathbf{q}) = \mathbf{q}$. Since $\mathbf{g} = \mathbf{k} + \mathbf{q}$, by exercise 10 (which wasn't assigned) we only need to show that KQ is closed.

To show closure, first note that Q is closed (since it is a normalizer) and that K is compact. Suppose that a sequence of matrices $k_j q_j$ converges in M to a matrix a. Since K is compact, there is a subsequence of k_j's that converges to k_∞. So without loss of generality, suppose that $k_j \to k_\infty$. Then $k_\infty^{-1} k_j q_j$ converges to $k_\infty^{-1} a$. Since $k_\infty^{-1} k_j$ converges to the identity, this means that $k_\infty^{-1} a$ is a limit point of Q, and hence is in Q, so $a \in KQ$.

2.5.12: $\mathbf{k}$ is the span of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and $K = SO(2)$. $\mathbf{q}$ is the space of upper-triangular traceless matrices, so Q is the space of upper-triangular matrices of determinant 1. Q has two connected components, one where the diagonal entries are positive and one where they are negative.