Lie Groups, Problem Set # 5 Solutions

Like last week, this week's problems were all from the book, namely Section 3.1, problems 4, 9, 11, 12, 13 and Section 3.2, 1.

Section 2.6

3.1.4 (π): For $X, Y \in \mathbf{g}$, let $\rho(X,Y) = Re(Tr(XY))$. If $a \in G$, then $\rho(Ad(a)X, Ad(a)Y) = Re(Tr(aXa^{-1}aYa^{-1})) = Re(Tr(aXYa^{-1})) = Re(Tr(XY)) = \rho(X,Y)$. Thus ρ is an Ad-invariant bilinear form on $\mathbf{g}$. What remains is showing that ρ is non-degenerate. This follows from the fact that all of the classical groups are invariant under Hermitian conjugation, which we showed in class. If $a \in G$, then $a^* \in G$, so if $X \in \mathbf{g}$, $X^* \in \mathbf{g}$. However, $\rho(X^*, X) = Re(Tr(X^*X))$ is the sum of the squared norms of the matrix entries of X, and is positive whenever X is non-zero.

3.1.9: Let C denote complex conjugation in $\mathbb{C}^n$. We already showed that real vector spaces can be viewed as complex vector spaces with extra structure, and that real operators are just complex operators that commute with C. So $Sp(n,\mathbb{R})$ is the set of $2n \times 2n$ complex matrices that commute with C and preserve an anti-symmetric bilinear form ϕ.

Define $\omega(X,Y) = i\phi(C(X),Y) = i\phi(\bar{X},Y)$. This is sesquilinear and Hermitian: $\omega(Y,X) = i\phi(\bar{Y},X) = -i\phi(X,\bar{Y}) = \overline{i\phi(\bar{X},Y)} = \overline{\omega}(X,Y)$. If $a \in Sp(n,\mathbb{R})$, then a preserves ϕ and commutes with C, so
$\omega(aX,aY) = i\phi(C(aX),aY) = i\phi(aC(X),aY) = i\phi(C(X),Y) = \omega(X,Y)$. The form ω has signature (n,n), since $\omega(C(X),C(X)) = i\phi(X,C(X)) = -i\phi(C(X),X) = -\omega(X,X)$. Since $Sp(n,\mathbb{R})$ preserves an anti-symmetric bilinear form and a hermitian form of signature (n,n), $Sp(n,\mathbb{R}) \subset Sp(n,\mathbb{C}) \cap SU(n,n)$.

For the converse, suppose that a preserves both ϕ and ω. Then a commutes with C, since the preservation of ω depends on the identity $aC(X) = C(aX)$. Thus a is a real matrix that preserves ϕ, so $Sp(n,\mathbb{C}) \cap SU(n,n) \subset Sp(n,\mathbb{R})$.

3.1.11. We will use the characterization of $\mathbf{g}$ from problem 13, below. (a) $\mathbf{sl}(n,\mathbb{C})$ are the traceless $n \times n$ matrices. With n^2 matrix entries and one linear condition, this has dimension $n^2 - 1$. (b) $\mathbf{so}(n,\mathbb{C})$ are the anti-symmetric matrices, spanned by $E_{jk} - E_{kj}$ with $j < k$. There are $\binom{n}{2} = n(n-1)/2$ such basis vectors. (c) Using the bilinear form $\begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$, $Sp(n,\mathbb{C})$ is the set of matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $D = -A^T$, $B = B^T$ and $C = C^T$. There are n^2 degrees of freedom for A, $n(n+1)/2$ for B, $n(n+1)/2$ for C, and none for D (which is determined by A), for a total of $2n^2 + n$

3.1.12. (a) $\mathbf{su}(n)$ is the set of traceless anti-Hermitian matrices, $i\mathbf{su}(n)$ is the set of traceless Hermitian matrices, so $\mathbf{su}(n) \oplus i\mathbf{su}(n)$ is the set of traceless matrices, which is $\mathbf{sl}(n, \mathbb{C})$. (b) $\mathbf{so}(n)$ is the set of real anti-symmetric matrices, so $\mathbf{so}(n) \oplus i\mathbf{so}(n)$ is the set of all anti-symmetric matrices, which is $\mathbf{so}(n, \mathbb{C})$ if we use the bilinear form 1_n. (c) $Sp(n)$ is the space of traceless matrices that preserve an anti-symmetric bilinear form ϕ on $\mathbb{C}^{2n}$ and commute with J, so $\mathbf{sp}(n)$ is the space of traceless matrices satisfying $\tilde\phi X = -X^T \tilde\phi$ and commuting with J. The space $i\mathbf{sp}(n)$ is matrices satisfying $\tilde\phi X = -X^T\tilde\phi$ and anti-commuting with J, since J is conjugate-linear. Thus $\mathbf{sp}(n) \oplus i\mathbf{sp}(n)$ is all matrices satisfying $\tilde\phi X = -X^T\tilde\phi$, regardless of J. But that is $\mathbf{sp}(n, \mathbb{C})$.

3.1.13 Recall that for $\mathbf{sl}(n, F)$ is all traceless matrices over the field F, and that $L(Aut(\phi))$ is all traceless matrices satisfying $\tilde\phi X = -X^T\tilde\phi$ for bilinear forms ϕ and $\tilde\phi X = -X^*\tilde\phi$ for sesquilinear forms. To see this last point, note that $\phi(aX, aY) = \phi(X, Y)$ boils down to $X^T a^T \tilde\phi a Y = X^T \tilde\phi Y$ for all X, Y (with ^{T} replaced by * for sesquilinear), hence $a^T\tilde\phi a = \tilde\phi$, hence $\tilde\phi a = (a^T)^{-1}\tilde\phi$. Taking a derivative as a passes through the origin gives $\tilde\phi X = -X^T\tilde\phi$.

(a,b) These are nearly trivial. Let $F = \mathbb{R}$ or $\mathbb{C}$. e^{tX} has determinant $e^{tTr(X)}$, and hence is in $SL(n, F)$ for all t if and only if $Tr(X) = 0$.

(c) As described on page 95, an $n \times n$ quaternionic matrix can be viewed as a complex matrix of the form $\begin{pmatrix} a & -\bar b \\ b & \bar a \end{pmatrix}$, so $gl(n, \mathbb{H})$ is all matrices of the form $\begin{pmatrix} X & -\bar Y \\ Y & \bar X \end{pmatrix}$. Note that the trace of this matrix is $2Re(tr(X))$. To sit in $\mathbf{sl}(n, \mathbb{H})$ we need the exponential to have determinant 1, which means $Re(tr(X)) = 0$.

(d) For $SO(p, q)$, we use the form given by $\tilde\phi = \begin{pmatrix} 1_p & 0 \\ 0 & -1_q \end{pmatrix}$, so the condition for $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ to be in $\mathbf{so}(p, q)$ is

$$\begin{pmatrix} A & B \\ -C & -D \end{pmatrix} = -\begin{pmatrix} A^T & -C^T \\ B^T & -D^T \end{pmatrix},$$

implying that A and D are anti-symmetric and $B = C^T$. In other words, $X = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}$ with A and D anti-symmetric. Note that the EXACT same description works for $SO(p, q, \mathbb{C})$.

(e) $SO(n, \mathbb{C})$ is just $SO(p, q, \mathbb{C})$ with $q = 0$. See (d).

(f) Since $\tilde{\phi} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, we are looking for real matrices $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with

$$\begin{pmatrix} -C & -D \\ A & B \end{pmatrix} = -\begin{pmatrix} C^T & -A^T \\ D^T & -B^T \end{pmatrix},$$

so B and C are symmetric and $A^T = -D$.

(g) This is the exact same calculation as (f).

(h) Now we have

$$\begin{pmatrix} A & B \\ -C & -D \end{pmatrix} = \begin{pmatrix} 1_p & 0 \\ 0 & -1_q \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = -\begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix}\begin{pmatrix} 1_p & 0 \\ 0 & -1_q \end{pmatrix} = \begin{pmatrix} -A^* & C^* \\ -B^* & D^* \end{pmatrix},$$

so A and D are anti-Hermitian and $C = B^*$. As a separate condition, the trace has to be zero. (We automatically have that the traces of A and D are pure imaginary, since the matrices are anti-Hermitian, but each one can have a non-zero trace as long as the sum of the two traces is zero.) Note that this is *not* the answer given in the book!! The book says that $B^* = -C$, which would mean that the whole matrix X would be anti-Hermitian (which is isn't).

(i) Typo alert! This problem is about the algebra $\mathbf{sp}(p,q)$, not the group $Sp(p,q)$. Also, when Rossman first says "complex matrices" in this problem, I think he means "quaternionic matrices", since the resulting pattern is $n \times n$ and not $2n \times 2n$. The second time he actually means "complex matrices". He also makes the same sign error as in (h).

The first part of this calculation, with quaternionic matrices, is identical to (h), with the result that $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $\bar{A}^T = -A$, $\bar{B}^T = C$ (not $-C$!) and $\bar{D}^T = -D$. This automatically means that the traces of A and D have no real part, so that the trace of the associated complex matrix is automatically zero.

Since a quaternionic matrix $a + jb$, with a and b complex, can be represented as a block matrix $\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}$, we can represent $A = \begin{pmatrix} A_1 & -\bar{A}_2 \\ A_2 & \bar{A}_1 \end{pmatrix}$, etc. yielding the complex form of $\mathbf{sp}(p,q)$.

(j) $SO^*(2n)$ preserves the anti-hermitian form given by the quaternionic matrix $j1_n$. This makes the criterion for being in the Lie algebra simply $jZ = -Z^*j$, which is equivalent to $Z^* = jZj$. Writing $Z = X + jY$, with X and Y complex, we have $X^* - jY^T = X^* - Y^*j = Z^* = jZj = -\bar{X} - j\bar{Y}$, so we must have X anti-symmetric and Y Hermitian. This gives the complex form $\begin{pmatrix} X & -\bar{Y} \\ Y & \bar{X} \end{pmatrix}$, as indicated in the book.

3.2.1: Sadly, this problem exhibits yet another sign error! Recall that the change-of-basis matrix for coordinates is the inverse of that for basis elements, in other words it's the change-of-basis for basis elements in the opposite direction. Our change-of-basis is equivalent to $e_j = (e'_j - ie'_{n+j})/\sqrt{2}$, $e_{n+j} = (e'_j + ie'_{n+j})/\sqrt{2}$. This implies that $\xi'_j = (\xi_j - i\xi_{n+j})/\sqrt{2}$ and $\xi'_{n+j} = (\xi_j + i\xi_{n+j})/\sqrt{2}$. That's not the right basis!

Instead, what we want is $\xi'_j = (\xi_j + \xi_{n+j})/\sqrt{2}$ and $\xi'_{n+j} = i(\xi_j - \xi_{n+j})/\sqrt{2}$, which is achieved by the change-of-basis with $e'_j = (e_j - ie_{n+j})/\sqrt{2}$ and $e'_{n+j} = (e_j + ie_{n_j})/\sqrt{2}$. Now let's apply this change of basis to the quadratic form.

(a) We compute $\xi'_j\eta'_j + \xi'_{j+n}\eta'_{j+n} = \frac{1}{2}[(\xi_j + \xi_{n+j})(\eta_j + \eta_{n+j}) - (\xi_j - \xi_{n+j})(\eta_j - \eta_{n+j}) = \xi_j\eta_{n+j} + \xi_{n+j}\eta_j$. Summing over j shows that the form in the problem is equal to the form in (1) or (2). Since our form is now represented by the identity matrix, the condition for being in the group reduces to $a^t a = 1$ (and $det(a) = 1$).

(b) $\xi'_j\bar{\eta}'_j = [\xi_j\bar{\eta}_j + \xi_j\bar{\eta}_{n+j} + \xi_{n+j}\bar{\eta}_j + \xi_{n+j}\bar{\eta}_{n+j}]/2$, while $\xi'_{n+j}\bar{\eta}'_{n+j} = [\xi_j\bar{\eta}_j - \xi_j\bar{\eta}_{n+j} - \xi_{n+j}\bar{\eta}_j + \xi_{n+j}\bar{\eta}_{n+j}]/2$, so the sum of the two is $\xi_j\bar{\eta}_j + \xi_{n+j}\bar{\eta}_{n+j}$. The change-of-basis we did in (a) changed the bilinear form but didn't change the Hermitian form.

For a to be unitary, we must have $a^*a = 1$, but to be in $SO(E)$ we must have $a^T a = 1$, so $a^T = a^*$, which implies that a is a real matrix. So $SO(E) \cap U(E) = SO(2n, \mathbb{R})$ or $SO(2n+1, \mathbb{R})$.

(c) For this part of the problem, the change-of-basis given in the book is correct. We only need to work on one block. Our original diagonal matrix had e_j and e_{n+j} as eigenvectors with eigenvalues ϵ and ϵ^{-1}. Set $\epsilon = \cos(2\pi\theta) + i\sin(2\pi\theta)$, for some (possibly complex) θ, which implies $\epsilon^{-1} = \cos(2\pi\theta) - i\sin(2\pi\theta)$, insofar as $\cos^2 + \sin^2 = 1$, even for complex arguments. Meanwhile, the matrix $\begin{pmatrix} \cos(2\pi\theta) & -\sin(2\pi\theta) \\ \sin(2\pi\theta) & \cos(2\pi\theta) \end{pmatrix}$ has eigenvalues $\cos(2\pi\theta) \pm i\sin(2\pi\theta)$ and eigenvectors $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \mp i \end{pmatrix}$. That is, we want $e_j = (e'_j - ie'_{n+j})\sqrt{2}$ and $e_{n+j} = (e'_j + ie'_{n+j})/\sqrt{2}$, which is exactly what is written.

Note: If we had used the change-of-basis that we used for parts (a) and (b), I believe that everything would have worked except that sin would have become $-\sin$. This can be corrected by changing the order of the basis $\{e'_k\}$ to $e'_{n+1}, e'_1, e'_{n+2}, e'_2, \ldots$. The main point that Rossman was making is correct, that the (complexification of the) torus of Theorem 1 on p96 is the same as the diagonal subgroup H that we have been studying for $SO(n, \mathbb{C})$.