Lie Groups, Problem Set # 6
Due Thursday, October 18

1) Consider the group $G = SO(p,q)$, obtained from the bilinear form ϕ on $\mathbb{R}^{p+q}$ with $\tilde{\phi} = \begin{pmatrix} 1_p & 0 \\ 0 & -1_q \end{pmatrix}$, where 1_p and 1_q are the $p \times p$ and $q \times q$ identity matrices. Writing matrices in block form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, find an explicit description of the Lie algebra $\mathbf{g} = \mathbf{so}(p,q)$, and of the subspaces $\mathbf{k}$ and $\mathbf{p}$. Show that $\mathbf{g}' = \mathbf{k} \oplus i\mathbf{p}$ is a Lie algebra. Find a group G' with $L(G') = \mathbf{g}'$, and show that G' is isomorphic (actually conjugate) to $SO(p+q)$. (It's not equal to $SO(p+q)$, since the matrices in G' aren't all real, but it's conjugate.)

The Lie algebra $\mathbf{g}$ is all matrices with $A^T = -A$, $D^T = -D$, and $B^T = C$. In other words, the upper left and lower right blocks are anti-symmetric and the rest of the matrix is symmetric. (See part (d) to problem 3.1.13 from last week, and restrict to real matrices) This means that $\mathbf{k}$ is the algebra of matrices with $B = C = 0$ (in other words, $\mathbf{k} = \mathbf{so}(p) \oplus \mathbf{so}(q)$) and $\mathbf{p}$ is the vector space of matrices with $A = D = 0$.

To show that G' is conjugate to $SO(n)$ with $n = p+q$, we merely show that $\mathbf{g}' = \mathbf{k} \oplus i\mathbf{p}$ is conjugate to $\mathbf{so}(n)$. Let a be a diagonal matrix whose first p entries are i and whose last q entries are 1. Then if $X = \begin{pmatrix} A & iB \\ iC & D \end{pmatrix} \in \mathbf{g}'$, with $B^T = C$, then $aXa^{-1} = \begin{pmatrix} A & -B \\ C & D \end{pmatrix}$, which is anti-symmetric and is the general form of an element of $\mathbf{so}(n)$. Since $a\mathbf{g}'a^{-1} = \mathbf{so}(n)$, $aG'a^{-1} = SO(n)$.

2) Next consider the group $G = SU(p,q)$, obtained from the Hermitian form ϕ on $\mathbb{C}^{p+q}$ with $\tilde{\phi} = \begin{pmatrix} 1_p & 0 \\ 0 & -1_q \end{pmatrix}$. As with Problem 1, find an explicit description of the Lie algebra $\mathbf{su}(p,q)$, of $\mathbf{k}$ and of $\mathbf{p}$. Show that $\mathbf{k} \oplus i\mathbf{p}$ is the Lie algebra of a group G' that is conjugate to $SU(p+q)$.

This is almost identical. $\mathbf{su}(p,q)$ is the set of complex traceless matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A^* = -A$, $D^* = -D$, and $B^* = C$. $\mathbf{k}$ is the matrices with $B = C = 0$ and $\mathbf{p}$ is the matrices with $A = D = 0$. In this case, $\mathbf{k} \oplus i\mathbf{p}$ is the space of all traceless anti-Hermitian matrices. In other words, $\mathbf{g}' = \mathbf{su}(n)$ and $G' = SU(n)$. (No conjugation necessary, but you can conjugate by the identity matrix if you really want to.)

3) What's next? $Sp(p,q)$, of course! (Same questions as problems 1 and 2, only with a slightly different group.)

As we saw last week, we want quaternionic matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $\bar{A}^T = -A$, $\bar{D}^T = -D$ and $\bar{B}^T = +C$. For $\mathbf{sp}(n)$ we would have had $\bar{B}^T = -C$ instead. So $\mathbf{k}$ is the matrices with $B = C = 0$ and $\mathbf{p}$ is the matrices with $A = D = 0$.

To understand $\mathbf{g}'$, we must remember that a quaternionic matrix $M_1 + jM_2$, with M_1 and M_2 complex, can be represented as a larger complex matrix $\begin{pmatrix} M_1 & -\bar{M}_2 \\ M_2 & -\bar{M}_1 \end{pmatrix}$. Multiplying this matrix by i yields $\begin{pmatrix} iM_1 & -i\bar{M}_2 \\ iM_2 & i\bar{M}_2 \end{pmatrix}$, which is *not* a quaternionic matrix. This means that G' is not a subgroup of $SL(n, \mathbb{H})$, much less a subgroup of $Sp(n)$. However, multiplying this matrix on the right or left by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ turns it back into a quaternionic matrix. Any $X \in \mathbf{g}'$ can be represented by a complex matrix of the form $\begin{pmatrix} A_1 & -\bar{A}_2 & iB_1 & -i\bar{B}_2 \\ A_2 & \bar{A}_1 & iB_2 & i\bar{B}_1 \\ iC_1 & -i\bar{C}_2 & D_1 & -\bar{D}_2 \\ iC_2 & i\bar{C}_1 & D_2 & \bar{D}_1 \end{pmatrix}$. Conjugating this by $a = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ gives $aXa^{-1} = \begin{pmatrix} A_1 & -\bar{A}_2 & iB_1 & i\bar{B}_2 \\ A_2 & \bar{A}_1 & iB_2 & -i\bar{B}_1 \\ iC_1 & -i\bar{C}_2 & D_1 & \bar{D}_2 \\ -iC_2 & -i\bar{C}_1 & -D_2 & \bar{D}_1 \end{pmatrix}$, which is a quaternionic matrix with $A' = A_1 + jA_2 = A$, $B' = iB_1 + j(iB_2)$, $C' = iC_1 + j(-iC_2)$ and $D' = D_1 + j(-D_2)$. Note that we have multiplied both components of B by i, while mutiplying one component of C by i and the other by $-i$. This converts the relation $\bar{B}^T = C$ to $\bar{B}'^T = -C'$, which makes aXa^{-1} an element of $\mathbf{sp}(n)$. We have also changed D, but in a legal way such that $\bar{D}'^T = -D'$.

4) Now consider $SO(2n, \mathbb{C})$ with the bilinear form ϕ with $\tilde{\phi} = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}$. Let H be the set of diagonal matrices in $SO(2n, \mathbb{C})$ and let $\mathbf{h}$ be the Lie algebra of H. Show that $\mathbf{h}$ consists of diagonal matrices with entries $(\lambda_1, \ldots, \lambda_n, -\lambda_1, \ldots, -\lambda_n)$, and that any $X \in \mathbf{g}$ that commutes with all of $\mathbf{h}$ is in $\mathbf{h}$. Find a basis for $\mathbf{g}$. Then decompose $\mathbf{g}$ into eigenspaces of $ad(\mathbf{h})$. (In other words, derive the D_n row in Table 3.6)

The condition for being in the Lie algebra is $\tilde{\phi}X = -X^T\tilde{\phi}$, which means that $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $B^T = -B$, $C^T = -C$, and $A^T = -D$. Put another way, if $j, k \leq n$, then $X_{j+n,k+n} = -X_{k,j}$, $X_{j+n,k} = -X_{k+n,j}$ and $X_{j,k+n} = -X_{k,j+n}$. For $\mathbf{h}$, we need $B = C = 0$ and A diagonal, so $D = -A$, which is exactly what we needed to show.

The $2n$ entries of a generic element $X \in \mathbf{h}$ are distinct. Since conjugation must take eigenspaces to eigenspaces, and since the eigenspaces of X are the coordinate directions, anything that commutes with X must itself be diagonal, and hence must be in $\mathbf{h}$.

Given our constraints, the obvious basis is: $\{E_{jk}-E_{n+k,n+j}, E_{j,n+k}-E_{k,n+j}, E_{n+j,k}-E_{n+k,j}\}$, where in the last two classes we want $j < k$. These are all eigenvectors of $ad(\mathbf{h})$, with eigenvalues $\lambda_j - \lambda_k$, $\lambda_j + \lambda_k$, and $-\lambda_j - \lambda_k$.

5) Repeat problem 4 for $SO(2n+1, \mathbb{C})$ with the bilinear form ϕ with $\tilde{\phi} = \begin{pmatrix} 0 & 1_n & 0 \\ 1_n & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, thereby deriving the B_n row of Table 3.6.

Letting $X = \begin{pmatrix} A & B & E \\ C & D & F \\ G & H & I \end{pmatrix}$, we have $A^T = -D$, $B^T = -B$ and $C^T = -C$ as before. In addition, we have $I = 0$, $E^T = -H$ and $F^T = -G$. In terms of matrix elements, this means that $X_{j+n,k+n} = -X_{k,j}$, $X_{j+n,k} = -X_{k+n,j}$ and $X_{j,k+n} = -X_{k,j+n}$ as before, and $X_{j,2n+1} = -X_{2n+1,n+j}$, $X_{j+n,2n+1} = -X_{2n+1,j}$, and $X_{2n+1,2n+1} = 0$. The diagonal subgroup is exactly as before, only with a 0 in the lower right corner. Since the eigenvalues $\{\pm\lambda_1, \pm\lambda_2, \ldots, \pm\lambda_n, 0\}$ are still generically distinct, the only matrix that commutes with a generic element of $\mathbf{h}$ must be diagonal, hence an element itself of $\mathbf{h}$.

In addition to the basis elements (and eigenvalues) from problem 4, we also have $E_{j,2n+1} - E_{2n+1,n+j}$ with eigenvalue λ_j and $E_{n+j,2n+1} - E_{2n+1,j}$ with eigenvalue $-\lambda_j$.

6) And do $C_n = Sp(n, \mathbb{C})$ to round things out.

Now we have matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A^T = -D$, $B^T = B$ and $C^T = C$. (See last week's HW.) In terms of matrix elements, $X_{jk} = -X_{n+k,n+j}$, $X_{n+j,k} = X_{n+k,j}$, and $X_{j,n+k} = X_{k,n+j}$. A basis for $\mathbf{g}$ is $E_{jk} - E_{n+k,n+j}$ with $j \neq k$, $E_{j,n+j}$, $E_{n+j,j}$, $E_{j,n+k} + E_{k,n+j}$ with $j < k$ and $E_{n+j,k} + E_{n+k,j}$ with $j < k$. The corresponding eigenvalues are $\lambda_j - \lambda_k$, $2\lambda_j$, $-2\lambda_j$, $\lambda_j + \lambda_k$, and $-\lambda_j - \lambda_k$.

The diagonal matrices have $D = -A$, hence of the same form as for $SO(2n, \mathbb{C})$. As before, the eigenvalues are generically distinct, so only a diagonal matrix can commute with all of $\mathbf{h}$.

Book problem 3.2.6: This problem refer to "G", which throughout the section means a complex classical group.

(a) If a is semi-simple, then it has eigenvalues and eigenspaces. We must pick a basis $\{\xi_k\}$ of eigenvectors and then consider the matrix p whose columns are the elements of our basis and show that this matrix is in G. Then $a = pdp^{-1}$, where d is diagonal. If a and p are in G, then so is d, so d is in H.

When $G = SL(n, \mathbb{C})$, there is nothing to show. The eigenvectors are linearly independent, and we can scale them so that $det(p) = 1$. In all the other cases, we must pick a basis such that $\phi(\xi_j, \xi_k) = \phi(e_j, e_k)$ and such that $det(p) = 1$. This will then imply that $p \in G$. We'll worry about the determinant last – it's pretty easy. For now, just concentrate on the eigenvectors.

If $a \in G$, then the eigenvectors of G have an orthogonality property: If $\lambda_j \lambda_k \neq 1$, then $\phi(\xi_j, \xi_k) = 0$. This is simply because $\phi(\xi_j, \xi_k) = \phi(a\xi_j, a\xi_k) = \lambda_j \lambda_k \phi(\xi_j, \xi_k)$. If the eigenvalues are $\epsilon_1^{\pm 1}, \epsilon_2^{\pm 1}, \ldots, \epsilon_n^{\pm 1}$ and possibly 1, with all of the eigenvalues distinct, then the eigenvector ξ_j with eigenvalue ϵ_j is orthogonal to everything but the eigenvector ξ_{n+j} with eigenvalue ϵ_j^{-1}. Since ϕ is non-degenerate, $\phi(\xi_j, \xi_{n+j})$ cannot be zero, and we can scale one of the vectors so that $\phi(\xi_j, \xi_{n+j}) = 1$. Which is what we wanted. (For $SO(2n+1, \mathbb{C})$, the last eigenvector ξ_{2n+1} is not orthogonal to itself. and we can scale it by a complex number so that $\phi(\xi_{2n+1}, \xi_{2n+1}) = 1$.)

Finally, the determinant. For $Sp(n, \mathbb{C})$ this is automatic, since the volume form is the n-th exterior power of the symplectic form ϕ. Anything that preserves ϕ preserves volume, and hence has determinant 1. For $SO(2n)$ or $SO(2n + 1)$, p preserving ϕ means that $det(p) = \pm 1$. If $det(p) = -1$, switch ξ_1 and ξ_{n+1}. This concludes the proof when a had distinct eigenvalues.

When a has repeated eigenvalues, we can still apply these arguments. Pick an arbitrary basis for theϵ eigenspace. Then pick a dual basis for the ϵ^{-1} eigenspace.

(b) This is identical to (a), only with the eigenvalues of $X \in \mathbf{g}$ coming in $\pm\lambda$ pairs instead of reciprocal pairs. The orthogonality relations still work, since the eigenvectors of X are the same as the eigenvectors of e^X.

(c) Recall that if matrices $X_1, \ldots, X_k$ are diagonalizable and commute, then it is possible to simultaneously diagonalize them. As before, each eigenvector with a set of eigenvalues is ϕ-orthogonal to all of the eigenspaces except the one with minus that set of eigenvalues. In other words, you can choose the basis ξ_j such that the eigenvalue of ξ_{n+j} for each $X \in \mathbf{h}$ is minus the eigenvalue for ξ_j. But then, after the change of basis, every element of $\mathbf{a}$ is in $\mathbf{h}$. [For what it's worth, I don't understand what the hint is driving at.]

(d) If A is connected and Abelian, then $A = \Gamma\mathbf{a}$, where $\mathbf{a}$ is an Abelian sub-algebra. By (c), $\mathbf{a}$ is conjugate to a sub-algebra of $\mathbf{h}$, so $A = \Gamma(\mathbf{a})$ is conjugate to a a subgroup of H. (I don't see what the hint has to do with it.)

(e) By (c), an Abelian sub-algebra consisting of semi-simple elements, then $\mathbf{a}$ must take the form $c\mathbf{h}_0c^{-1}$ for some $c \in G$ and some sub-algebra $\mathbf{h}_0$ of $\mathbf{h}$. But this is a sub-algebra of $c\mathbf{h}c^{-1}$, so if $\mathbf{a}$ is maximal, it must equal $c\mathbf{h}c^{-1}$.

(f) Let $\mathbf{a} = L(A)$. Then $\mathbf{a}$ consists of semi-simple elements since a matrix that diagonalizes e^{tX} for t small also diagonalizes X. Now $\mathbf{a}$ must be maximal, since if $\mathbf{a}^+$ is an Abelian and semi-simple extension of $\mathbf{a}$, then $\Gamma(\mathbf{a}^+)$ is an Abelian and semi-simple extension of A. By (e), $\mathbf{a} = c\mathbf{h}c^{-1}$, so $A = cHc^{-1}$.

(g) For a specific counter-examples, consider $SO(3)$ with the bilinear form. The 4-element group generated by the rotations by π around the three coordinate axes does not have any vectors that are fixed by the entire group. Since evey element of H fixes e_3, our 4-element group is not conjugate to a subgroup of H,

(h) The simplest example is in $SL(2, \mathbb{C})$. The algebra $\mathbf{a}$ generated by $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is maximal, since there are no 2-dimensional Abelian subalgebras of $\mathbf{sl}(2, \mathbb{C})$. But there's no way for a non-diagonalizable matrix to be conjugate to an element of $\mathbf{h}$! Exponentiating, consider the Abelian group consisting of matrices of the form $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$. It is maximal Abeilian since it is connected and its algebra is maxima Abelianl. But it contains non-diagonalizable elements, so it can't be conjugate to a subgroup of H.

3.2.9 This is in some sense the compact analog of problem 6. Let T be a maximal torus, and let $\mathbf{t}$ be its Lie algebra. Since T is compact, each element of $\mathbf{t}$ must be diagonalizable with pure imaginary eigenvalues. (Or else $\exp(tX)$ would be unbounded). We have already proven that elements of the compact groups can be diagonalized by elements of G (the analog of problem 6a). So simultaneously diagonalize the elements of $\mathbf{t}$ to get that $\mathbf{h}$ is conjugate to a subgroup of the diagonal subgroup. Since T is maximal and $T = \Gamma(\mathbf{t})$, $\mathbf{t}$ must be conjugate to the entire diagonal subgroup, so T is conjugate to a Cartan subgroup.