Lie Groups, Problem Set # 7
Due Tuesday, October 30

In this problem set we're going to classify the irreducible root systems. It's a long list of problems, but you can skip the starred exercises that we already did in class.

Recall from lecture (or p119 of the book) that an abstract root system is a subset Φ of a Euclidean space L such that:

1. Φ is a finite set that spans L and does not contain 0. (The elements of Φ are called roots.)
2. For $\alpha \in \Phi$, the only multiples of $\alpha \in \Phi$ are $\pm\alpha$.
3. For $\alpha \in \Phi$, the reflection along α (given by the formula $\beta \mapsto \beta - \frac{2\alpha\cdot\beta}{\alpha\cdot\alpha}\alpha$) maps Φ to itself.
4. For any two $\alpha, \beta \in \Phi$, $2\frac{\alpha\cdot\beta}{\alpha\cdot\alpha}$ is an integer.

1*) Show that the angle between any two roots is either 0, 30, 45, 60, 90, 120, 135, 150 or 180 degrees. Further, show that the two roots have the same length if the angle is 0, 60, 120 or 180 degrees, that the ratio of their lengths is $\sqrt{2}$ if the angle is 45 or 135 degrees, and that the ratio of their lengths is $\sqrt{3}$ if the angle is 30 or 150 degrees.

Since $2(\alpha\cdot\beta)/(\alpha\cdot\alpha)$ and $2(\alpha\cdot\beta)/(\beta\cdot\beta)$ are both integers, the product $4\cos^2(\theta) = 4(\alpha\cdot\beta)^2/[(\alpha\cdot\alpha)(\beta\cdot\beta)]$ is an integer. But $\cos^2(\theta) \leq 1$, so this product is either 0, 1, 2, 3 or 4. WLOG, assume that $|\alpha| \geq |\beta|$. Then $|\alpha|^2/|\beta|^2$ is a ratio of small integers whose product is 1, 2, 3, or 4.

If the product of integers is 0, then θ is 90 degrees, and we can't say anything about lengths. (Our "ratio" is 0/0.)

If the product of integers is 1, then θ is 60 or 120 degrees. Furthermore, the ratio of integers is 1 (since either both integers are 1 or both are -1), so $|\alpha| = |\beta|$.

If the product is 2, then $\cos^2(\theta) = 1/2$ and θ is 45 or 135 degrees. The ratio must be $2 = 2/1 = (-2)/(-1)$, so $|\alpha| = \sqrt{2}|\beta|$.

If the product is 3, then $\cos^2(\theta) = 3/4$, so θ is 30 or 150 degrees. The ratio is $3 = 3/1 = (-3)/(-1)$, so $|\alpha| = \sqrt{3}|\beta|$.

If the product is 4, then $\cos^2(\theta) = 1$, so the two roots are colinear. By axiom 2, this means they are either equal ($\theta = 0$ and $|\alpha| = |\beta|$) or negatives ($\theta = \pi$ and $|\alpha| = |\beta|$).

2) Pick a hyperplane through the origin that does not intersect Φ. We call the roots on one side of this hyperplane positive, and the roots on the other side negative. A positive root is *simple* if it cannot be written as the sum of two other positive roots. Show that the angle between any two simple roots must be at least 90 degrees. [This was mostly done in class. Finish the proof.]

If θ is 30, 45 or 60 degrees and $|\alpha| \geq |\beta|$, we must have $2(\alpha \cdot \beta)/(\alpha \cdot \alpha) = 1$. But then the reflection of β along α is $\gamma = \beta - \alpha$. Since either γ or $-\gamma$ is a positive root, either $\alpha = (-\gamma) + \beta$ or $\beta = \gamma + \alpha$ is a sum of positive roots and so isn't simple. Contradiction. [The pictures of the reflections are slightly different for the three cases, but the argument is the same.]

3) For the A_n, B_n, C_n and D_n root lattices, pick the hyperplane $\ell_1 + \epsilon\ell_2 + \epsilon^2\ell_3 + \cdots \epsilon^{n-1}\ell_n = 0$ for sufficiently small ϵ. Show that the positive roots (with the obvious overall sign choice) are exactly the same as those listed in the book. What are the simple roots? [Just list the simple roots – we did the rest in class]

For A_{n-1}, the roots are $\lambda_j - \lambda_k$, the positive roots have $j < k$, and the simple roots have $k = j + 1$.

For B_n, the roots are $\lambda_j - \lambda_k$, $\lambda_j + \lambda_k$, $-\lambda_j - \lambda_k$, λ_j and $-\lambda_j$. The positive roots are $\lambda_j - \lambda_k$ with $j < k$, $\lambda_j + \lambda_k$, and λ_j. The simple roots are $\lambda_j - \lambda_{j+1}$ and λ_n.

For C_n, the roots are $\lambda_j - \lambda_k$, $\lambda_j + \lambda_k$, $-\lambda_j - \lambda_k$, $s\lambda_j$ and $-s\lambda_j$. The positive roots are $\lambda_j - \lambda_k$ with $j < k$, $\lambda_j + \lambda_k$, and $2\lambda_j$. The simple roots are $\lambda_j - \lambda_{j+1}$ and $2\lambda_n$.

For D_n, the roots are $\lambda_j - \lambda_k$, $\lambda_j + \lambda_k$, and $-\lambda_j - \lambda_k$. The positive roots are $\lambda_j - \lambda_k$ with $j < k$ and $\lambda_j + \lambda_k$. The simple roots are $\lambda_j - \lambda_{j+1}$ and $\lambda_{n-1} + \lambda_n$.

We draw a graph, called a Dynkin diagram, to describe the geometry of the simple roots. Draw one dot for each simple root. Draw single, double, or triple lines connecting two simple roots if the angle between them is 120, 135 or 150 degrees. (Don't draw a line for a 90 degree angle). For the double and triple lines, draw an arrow on the line pointing from the longer root towards the shorter root.

Notation: we call a vertex that is connected to more than two other vertices a *branch point*.

In a root system $\Phi = \Phi_1 \oplus \Phi_2$ is a direct sum of two lower-dimensional root systems, then the simple roots of Φ will be the union of the simple roots of Φ_1 and the simple roots of Φ_2. This will lead to a disconnected Dynkin diagram. If the root system is irreducible, meaning that it isn't a direct sum, then the resulting Dynkin diagram is connected.

4*) From your results in (3), draw the Dynkin diagrams for A_n, B_n, C_n and D_n. Note that B_2 and C_n are given by the same diagram, and that D_3 has the same diagram as A_3. In fact, the Lie algebras are isomorphic. For this reason we usually only consider B_n for $n > 1$, C_n for $n > 2$ and D_n for $n > 3$.

(I don't know how to generate nice pictures in TeX, so I'll use the following conventions. $-$ is a single line, $=>$ is a double line, and $=> -$ is a triple line.)

The diagram for A_n is $\alpha_1 - \alpha_2 - \cdots - \alpha_{n-1}$.

The diagram for B_n is $\alpha_1 - \alpha_2 - \cdots - \alpha_{n-1} => \alpha_n$, since $\lambda_{n-1} - \lambda_n$ is bigger than λ_n.

The diagram for C_n is $\alpha_n => \alpha_{n-1} - \cdots - \alpha_1$, since $2\lambda_n$ is bigger than $\lambda_n - \lambda_{n-1}$.

The diagram for D_n has a chain $\alpha_1 - \cdots - \alpha_{n-2}$, with $\alpha_{n-2} = \lambda_{n-2} - \lambda_{n-1}$ then connected to both $\alpha_{n-1} = \lambda_{n-1} - \lambda_n$ and $\alpha_n = \lambda_{n-1} + \lambda_n$. Note that α_{n-1} and α_n are orthogonal.

The classification of irreducible root systems proceeds by showing that various sub-diagrams can't occur.

5*) Show that if α and β are connected by a triple line, then α and β are not connected to any other vertices. In other words, the only connected Dynkin diagram with a triple line has only two vertices. This corresponds to the rank-2 group G_2. [Hint: I see two ways to do this, and they're in some sense equivalent. One is to show that there aren't any directions that make at least a 120 degree angle with α and at least a 90 degree angle with β. The other is to assume that there is a root γ with this property and find a linear combination of α, β and γ whose squared length is non-positive, which is of course a contradiction.]

If we had $\alpha => -\beta - \gamma$, then $(\alpha + 2\beta + \gamma) \cdot (\alpha + 2\beta + \gamma) \leq 0$, with equality iff α and γ are orthogonal. If we had $\alpha => -\beta = \gamma$ or $\alpha => -\beta = -\gamma$, then after scaling γ to have the same size as β the squared norm would be negative.

We rule out $\alpha - \; <= \beta - \gamma$, etc similarly by looking at the squared norm of $3\alpha + 2\beta + \gamma$.

6*) Show that a connected diagram can have at most one double line. [Hint: assume that you have two double lines and find a linear combination of the roots with negative squared length.]

If we have $\alpha_1 <= \alpha_2 - \cdots - \alpha_{n-1} => \alpha_n$ and $|\alpha_1| = 1$, then $(\sum \alpha_j) \cdot (\sum \alpha_j) = 1 + 2 + 2 + \cdots + 2 + 1 - 2 - 2 - 2 - \cdots - 2 = 0$ (or less than 0 if there are any additional cross-terms). If either of the double lines point in instead of out, replace

α_1 or α_2 with $\alpha_1/2$ or $\alpha_n/2$ in the sum.

7) Show that a diagram containing a double line cannot have a branch point.

If we have $\alpha_1 <= \alpha_2 - \cdots - \alpha_{n-2}$, with α_{n-2} connected to α_{n-1} and α_n, then $\alpha_1 + \cdots + \alpha_{n-2} + (\alpha_{n-1} + \alpha_n)/2$ has squared norm $1+2+\cdots+2+\frac{1}{2}+\frac{1}{2}-2-2-\cdots-2 = 0$ (or less if a line is double or if there are cross-terms).

If the double line points the other way, replace α_1 with $\alpha_1/2$.

8*) Show that a diagram with no double lines cannot have any loops.

If there is a closed loop $\alpha_1 - \alpha_2 - \cdots - \alpha_n$, with α_n then connected back to α_1, then $(\sum_j \alpha_j) \cdot (\sum_j \alpha_j) = 0$ if there are no additional lines. (Imagine $\alpha_j = \lambda_j - \lambda_{j+1}$ and $\alpha_n = \lambda_n - \lambda_1$.)

9*) Show that a diagram with a double line cannot have any loops.

If we had a loop containing a double line, then the roots on opposite sides of the double line would have to have the same length, since they're connected by single lines, which contradicts the properties of double lines.

10) Show that, with one exception, you can't have a diagram with a double line that extends in both directions. Specifically, you can't have α connected to β, β connected to γ with a double line (pointing either way), γ connected to δ, and δ connected to ϵ. (If you eliminate the ϵ then the resulting graph with 4 vertices *is* possible, and corresponds to the group F_4.)

We are trying to rule out $\alpha - \beta => \gamma - \delta - \epsilon$. Let $r = \alpha + 2\beta + 3\gamma + 2\delta + \epsilon$, and suppose that $|\epsilon| = 1$. Then $r \cdot r = 2 + 8 + 9 + 4 + 1 - 4 - 12 - 6 - 2 = 0$. To rule out $\alpha - \beta <= \gamma - \delta - \epsilon$, look at $2\alpha + 4\beta + 3\gamma + 2\delta + 1\epsilon$ instead.

11) Draw all the possible diagrams that involve a double or triple line.

The only diagram with a triple line is $\alpha => -\beta$, which corresponds to the exceptional algebra (and group) G_2.

The diagrams with a double line are $\alpha_1 - \alpha_2 => \alpha_3 - \alpha_4$, corresponding to F_4, plus the already-listed diagrams for B_n and C_n. (By problem 10, a double line either extends 1 in both directions, or 0 in one direction and an arbitrary distance in the other. Extending by 0 in both directions is $B_2 = C_2$.)

We now consider the diagrams where all of the lines are single. These are called the "simply laced" diagrams. The remaining questions are all about simply laced diagrams.

12) Show that any simply laced diagram has at most three ends. In particular, there can be at most one branch point, and it can only be a 3-way intersection.

Suppose you have two or more branches. Then there exists a sub-diagram consisting of a chain $\alpha_2 - \alpha_3 - \cdots - \alpha_{n-2} - \alpha_{n-1}$,with α_1 connected to α_3 (not to α_2) and α_n connected to α_{n-2} (not to α_{n-1}). The case $n = 5$ is a single branch point with 4 (or more) legs. All roots have the same length, which we take to be 1. Let $r = \alpha_1 + \alpha_2 + 2\sum_{j=3}^{n-2} \alpha_j + \alpha_{n-1} + \alpha_n$. Then $r \cdots r = 1+1+4+4+\cdots+4+1+1-2-2-4-4-4-\cdots-4-2-2 = 0$. This can be realized explicitly with $\alpha_1 = \lambda_1 - \lambda_2$, $\alpha_2 = -\lambda_1 - \lambda_2$, $\alpha_j = \lambda_j - \lambda_{j+1}$, $\alpha_{n-1} = \lambda_{n-1} - \lambda_n$ and $\alpha_n = \lambda_{n-1} + \lambda_n$. You can check that $r = 0$.

What remains is to limit the size of the ends.

13) Show that at least one of the branches coming from a branch point has length 1.

Suppose that we have a branch point α connected to three branches of length at least 2, with $\beta_{1,2,3}$ connected to α and γ_j connected to β_j. Let $r = 3\alpha + 2\sum \beta_j + \sum \gamma_j$. Then $r \cdot r = 9 + 4 + 4 + 4 + 1 + 1 + 1 - 3(6) - 3(2) = 0$.

14) Show that you cannot have two of the three branches having length 3 or more.

Suppose we had a chain $\alpha_1 - \cdots - \alpha_7$ with α_4 also connected to β. Let $r = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 + 2\beta$. Then $r \cdot r = 1 + 4 + 9 + 16 + 9 + 4 + 1 + 4 - 2 - 6 - 12 - 12 - 6 - 2 - 8 = 0$.

15) Show that if there is a branch point and two of the branches have length 1 and 2, then the third branch has length at most 4.

Let's introduce some terminology. The E_{n+1} diagram is a chain $\alpha_1 - \cdots - \alpha_n$ with β connected to α_3. We need to show that E_9 is impossible. Let $r = 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8 + 3\beta$. Then $r \cdot r = 4 + 16 + 36 + 25 + 16 + 9 + 2 + 1 + 9 - 8 - 24 - 30 - 20 - 12 - 6 - 2 - 18 = 0$.

16) Write down all of the simply laced Dynkin diagrams.

These are A_n, D_n, E_6, E_7 and E_8. For the E_n series we can't have $n \geq 9$, and E_5 is just D_5.

Where did I get those crazy linear combinations?!

It's not black magic. In each case we're trying to find a linear combination $r = \sum c_j \alpha_j$ whose square is zero, leading to a contradiction. For simply laced diagrams, $r \cdot r = \sum_j c_j^2 - \sum_{\text{edges}} c_j c_k$. Minimizing with respect to c_j means that c_j must equal half the sum of the coefficients of vertices that touch α_j. This completely determines r (up to scale). Pick the coefficient of an endpoint, and the rest follow, one by one.

In particular, the coefficients along chains must form an arithmetic progression; if the endpoint has coefficient x, then the next point has coefficient $2x$, the next has coefficient $3x$, etc.

The diagrams with double and triple lines are only slightly more complicated. In general, the value of c_j that minimizes $r \cdot r$ is $-\frac{1}{2}\sum_\beta c_\beta \frac{2\alpha_j\cdot\beta}{\alpha_j\cdot\alpha_j}$. If α_j is the longer root of a double or triple line, $2\alpha_j \cdot \beta/\beta \cdot \beta = -1$, just as with a single line. If α_j is the shorter root of a double or triple line, then we multiply c_β by 2 or 3 in its contribution to c_j.

Combining problem 16 with problem 11 gives a complete classification of possible Dynkin diagrams. The ones that aren't A_n or D_n are called E_6, E_7 and E_8, with the subscript indicating how many dots are in the diagram (aka the rank of the group).

The only thing that's missing is the actual Lie theory! The relevant theorems are:

Theorem 1 *Let G be a simple Lie group, let H be a maximal Abelian subgroup consisting of semi-simple elements, and let Φ be the resulting root system. Then the Dynkin diagram for G is connected and does not depend on the choice of H (since all Cartan subgroups are conjugate) and does not depend on the hyperplane used to define positivity.*

Theorem 2 *The same thing goes for simple Lie algebras.*

Theorem 3 *Two simple Lie algebras are isomorphic if and only if they generate the same Dynkin diagram.*

Theorem 4 *All simple Lie algebras are of the form A_n, B_n, C_n, D_n E_6, E_7, E_8, F_4 or G_2. All semi-simple Lie algebras are direct sums of simple Lie algebras. All simply connected semi-simple Lie groups are obtained by exponentiation from semi-simple Lie algebras.*