M346 Final Exam, December 11, 2013

1. Basic row operations. Consider the matrix $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 8 \\ 2 & 5 & 8 & 12 \end{pmatrix}$.

a) Find a basis for the null space of A.

A row-reduces to $\begin{pmatrix} 1 & 0 & -1 & -4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, which gives the equations

$$\begin{aligned} x_1 &= x_3 + 4x_4 \\ x_2 &= -2x_1 - 4x_4 \\ x_3 &= x_3 \\ x_4 &= x_4 \end{aligned}$$

so our basis is $\{(1, -2, 1, 0)^T, (4, -4, 0, 1)^T\}$.

b) Find a basis for the column space of A.

Since there are pivots in the first two columns, we want the first two columns of A, namely $\{(1, 1, 2)^T, (2, 3, 5)^T\}$. (Strictly speaking, ANY two columns of A will be linearly independent and form a basis, but the standard algorithm gives the first two columns.)

c) Find all solutions to $A\mathbf{x} = \begin{pmatrix} 5 \\ 9 \\ 14 \end{pmatrix}$.

Now our row-reduction yields $\begin{pmatrix} 1 & 0 & -1 & -4 & | & -3 \\ 0 & 1 & 2 & 4 & | & 4 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$, which gives the general solution $\mathbf{x} = \begin{pmatrix} -3 \\ 4 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 4 \\ -4 \\ 0 \\ 1 \end{pmatrix}$.

2. Bases and coordinates. Let V be the space of periodic functions with period 2π, and let E_{-9} be the eigenspace of the operator d^2/dx^2 with eigenvalue -9. E_{-9} is 2-dimensional. One obvious basis is $\mathcal{B} = \{\cos(3x), \sin(3x)\}$. Another basis is $\mathcal{D} = \{e^{3ix}, e^{-3ix}\}$.

a) Compute the change-of-basis matrices $P_{\mathcal{BD}}$ and $P_{\mathcal{DB}}$. Since $\mathbf{d}_1 = \mathbf{b}_1 + i\mathbf{b}_2$ and $\mathbf{d}_2 = \mathbf{b}_1 - i\mathbf{b}_2$, we have $P_{\mathcal{BD}} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$ and $P_{\mathcal{DB}} = P_{\mathcal{BD}}^{-1} = \begin{pmatrix} 1/2 & -i/2 \\ 1/2 & i/2 \end{pmatrix}$.

b) If $f(x) = 2\cos(3x) + 3\sin(3x)$, what are $[\mathbf{f}]_{\mathcal{B}}$ and $[\mathbf{f}]_{\mathcal{D}}$?

$[\mathbf{f}]_{\mathcal{B}} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $[\mathbf{f}]_{\mathcal{D}} = P_{\mathcal{D}\mathcal{B}}[\mathbf{f}]_{\mathcal{B}} = \begin{pmatrix} \frac{2-3i}{2} \\ \frac{2+3i}{2} \end{pmatrix}$.

c) Let L be the linear operator $L(f) = f + \frac{df}{dx}$. Compute the matrices $[L]_{\mathcal{B}}$ and $[L]_{\mathcal{D}}$ of L in the $\mathcal{B}$ and $\mathcal{D}$ bases.

It's easy to see that $\mathbf{d}_{1,2}$ are eigenvectors of L with eigenvalues $1 \pm 3i$, so $[L]_{\mathcal{D}} = \begin{pmatrix} 1+3i & 0 \\ 0 & 1-3i \end{pmatrix}$. We can compute $[L]_{\mathcal{B}}$ either with a change-of-basis, or by applying L directly to $\mathbf{b}_{1,2}$. Either way, the answer is $\begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix}$.

3. Computing eigenvalues and eigenvectors.

a) Find the characteristic polynomial of $A = \begin{pmatrix} 1 & 2 & 3 \\ -2 & 1 & 3 \\ -3 & -3 & 1 \end{pmatrix}$. You do **not** have to compute the eigenvalues or eigenvectors!

This works out to $(\lambda - 1)^3 + 22(\lambda - 1) = \lambda^3 - 3\lambda^2 + 25\lambda - 23$. (For what it's worth, the eigenvalues are 1 and $1 \pm \sqrt{22}i$.)

b) If a matrix B has characteristic polynomial $p_B(\lambda) = \lambda^4 - 2\lambda^3 + 2\lambda^2$, what are the eigenvalues of B, and with what (algebraic) multiplicity?

This factors as $\lambda^2(\lambda^2 - 2\lambda + 2)$ with roots 0 (with multiplicity 2) and $1 \pm i$ (with multiplicity 1).

c) Find the eigenvalues of the matrix $C = \begin{pmatrix} 5 & 4 & 6 & 1 & 0 \\ -1 & 1 & 2 & 5 & 3 \\ 0 & 0 & 2 & -2 & 6 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & -3 & 1 & -7 \end{pmatrix}$. Indicate the algebraic multiplicity and the geometric multiplicity of each eigenvalue. However, you do **NOT** need to compute the eigenvectors. (This is a "tricks of the trade" problem. For heaven's sakes, don't do it by brute force.)

This is block triangular. The upper left block has 3 as a double root, but only has one eigenvector. So $m_a(3) = 2$ and $m_g(3) = 1$. The lower right block has column sum 1, determinant zero, and trace -3, so its eigenvalues must be 1, 0, and -4. All of these have algebraic multiplicity 1, so all of them have geometric multiplicity 1.

4. Using diagonalization. The eigenvalues of the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

are 1, -1, i, and $-i$.

a) Find the eigenvectors of A.

$(1,1,1,1)^T$, $(1,-1,1,-1)^T$, $(1,i,-1,-i)^T$ and $(1,-i,-1,i)^T$, respectively. These form the columns of P in $A = PDP^{-1}$. Since these vectors are orthogonal and all have norm 2, $P/2$ is unitary, so finding P^{-1} is easy: $P^{-1} = \frac{1}{4}\overline{P^T}$.

b) Compute A^{2013}.

Since all four eigenvalues are fourth roots of 1 and 2013 is one more than a multiple of 4, $\lambda^{2013} = \lambda$, so $A^{2013} = PD^{2013}P^{-1} = PDP^{-1} = A$.

c) Compute $e^{\pi A}$.

Since $e^{i\pi} = -1$, $e^{\pi A} = Pe^{\pi D}P^{-1}$ equals

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & i & -i \\ 1 & -1 & i & -i \\ 1 & -1 & -i & i \end{pmatrix} \begin{pmatrix} e^{\pi} & 0 & 0 & 0 \\ 0 & e^{-\pi} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \\ 1 & i & -1 & -i \end{pmatrix}$$

This works out to

$$\frac{1}{4}\begin{pmatrix} e^{\pi}+e^{-\pi}-2 & e^{\pi}-e^{-\pi} & e^{\pi}+e^{-\pi}+2 & e^{\pi}-e^{-\pi} \\ e^{\pi}-e^{-\pi} & e^{\pi}+e^{-\pi}-2 & e^{\pi}-e^{-\pi} & e^{\pi}+e^{-\pi}+2 \\ e^{\pi}+e^{-\pi}+2 & e^{\pi}-e^{-\pi} & e^{\pi}+e^{-\pi}-2 & e^{\pi}-e^{-\pi} \\ e^{\pi}-e^{-\pi} & e^{\pi}+e^{-\pi}+2 & e^{\pi}-e^{-\pi} & e^{\pi}+e^{-\pi}-2 \end{pmatrix}$$

5. Linearization. The motion of a pair of coupled pendula are described by the equations

$$\begin{aligned} \frac{d^2x_1}{dt^2} &= -\sin(x_1) - \sin(x_1 - x_2) \\ \frac{d^2x_2}{dt^2} &= \sin(x_1 - x_2) - \sin(x_2) \end{aligned}$$

Here x_1 and x_2 are the angles of each pendulum relative to vertical, so $x_1 = 2\pi$ means the same thing as $x_1 = 0$. The four fixed points are $(0,0)^T$, $(0,\pi)^T$, $(\pi,0)^T$, and $(\pi,\pi)^T$.

a) For each fixed point, find a system of linear differential equations that approximate the behavior of this system near the fixed point.

Our matrix $df = \begin{pmatrix} -\cos(x_1) - \cos(x_1 - x_2) & \cos(x_1 - x_2) \\ \cos(x_1 - x_2) & -\cos(x_1 - x_2) - \cos(x_2) \end{pmatrix}$.

Near each fixed point we have $\mathbf{y} = \mathbf{x} - \mathbf{a}$ and $\ddot{\mathbf{y}} \approx A\mathbf{y}$, where the matrix A is:

$\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$ for $\mathbf{a} = (0,0)^T$.

$\begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}$ for $\mathbf{a} = (0,\pi)^T$.

$\begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}$ for $\mathbf{a} = (\pi,0)^T$.

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ for $\mathbf{a} = (\pi,\pi)^T$.

b) At each fixed point, indicate how many stable, neutrally stable, and unstable modes the linearlized system has.

For 2nd order systems, we never have stable modes, just unstable with positive eigenvalues and neutral with negative eigenvalues. Near $(0,0)$ the eigenvalues are -3 and -1 and we have 2 neutral modes. Near $(0,\pi)$, and near $(\pi,0)$, we have one unstable and one neutral mode, with eigenvalues $1 \pm \sqrt{2}$. Near (π,π) we also have one unstable and one neutral mode, with eigenvalues ± 1.

6. Inner products.

a) Let $V = \mathbb{R}_2[t]$ be the space of quadratic polynomials. Given this space the inner product $\langle \mathbf{f} | \mathbf{g} \rangle = \int_1^3 f(t)g(t)dt$. Find an orthogonal basis for V.

We apply Gram-Schmidt to our obvious basis of $\mathbf{x_1} = 1$, $\mathbf{x_2} = t$ and $\mathbf{x_3} = t^2$ to get $\mathbf{y_1} = 1$, $\mathbf{y_2} = t - 2$ and $\mathbf{y_3} = (t-2)^2 - 1/3 = t^2 - 4t + 11/3$.

b) For obscure reasons, a researcher believes that his data should take the form $y = a + bx^2 + ce^x$. Set up a least-squares problem to compute the coefficients a, b and c given the data points $(-1,4)$, $(0,0)$, $(1,1)$, $(2,1)$, and $(3,2)$. Explain how you would solve the least-squares problem to find the coefficients. However, **you do not need to actually do the computation – it's way to grungy for an exam**.

Plugging in the values of (x,y) gives 5 equations in the 3 unknowns:

$$\begin{pmatrix} 1 & 1 & e^{-1} \\ 1 & 0 & 1 \\ 1 & 1 & e \\ 1 & 4 & e^2 \\ 1 & 9 & e^3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 1 \\ 1 \\ 2 \end{pmatrix},$$

which we abbreviate as $A\mathbf{x} = \mathbf{b}$. There are no exact solutions. We find the least-squares solutions by solving $A^T A\mathbf{x} = A^T\mathbf{b}$.

7. Fourier series and partial differential equations. Consider the differential equation

$$\frac{\partial f(x,t)}{\partial t} = 3f(x,t) + f'(x,t) + f'''(x,t),$$

where $f(x,t)$ is a periodic function (of x) with period 2π that evolves in time.

a) Find the eigenvalues and eigenvectors of the operator $A = 3 + \frac{d}{dx} + \frac{d^3}{dx^3}$ on the space of periodic functions of period 2π. (This is easier than it looks, and does NOT require you to understand 3rd order differential equations.)

Since the vectors e^{inx} are eigenvectors of d/dx (with eigenvalue in), they are also eigenvectors of $(d/dx)^3$ (with eigenvalue $(in)^3 = -in^3$) and $3+d/dx+(d/dx)^3$ with eigenvalue $3 + in - in^3$.

b) Write down the general solution to the differential equation.

$$f(x,t) = \sum_{n=-\infty}^{\infty} c_n e^{inx} e^{(3+in-in^3)t}.$$

We can get the coefficients c_n from the initial conditions, since $f(x,0) = \sum_n c_n e^{inx}$.

c) Find a solution with initial condition $f(x,0) = 1 + 2\cos(x) + 6\cos(2x)$.

There are only 5 nonzero coefficients, namely $c_0 = c_{-1} = c_1 = 1$ and $c_2 = c_{-2} = 3$. (See problem 2 for a reminder of how to convert sines and cosines to complex exponentials.) Notice that $n - n^3 = 0$ for $n = 0, 1, -1$, that $2 - 2^3 = -6$ and $(-2) - (-2)^3 = 6$. Our solution is then

$$f(x,t) = e^{3t}\left[1 + e^{ix} + e^{-ix} + 3e^{2ix-6it} + 3e^{-2ix+6it}\right]$$

This simplifies to $e^{3t}(1 + 2\cos(x) + 6\cos(2(x-3t)))$.