M346 First Midterm Exam Solutions, September 26, 2013

1) Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 3 & 5 \\ 3 & 6 & 7 & 10 \end{pmatrix}.$$

Find the reduced row-echelon form A_{rref}, find a basis for the null space of A, and find a basis for the column space of A.

The matrix row-reduces to

$$A_{rref} = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

from which we can read off the basis for the null space:

$$\begin{aligned} x_1 &= -2x_2 - x_4 \\ x_2 &= \quad x_2 \\ x_3 &= \qquad\quad - x_4 \\ x_4 &= \qquad\quad x_4 \end{aligned}$$

so our basis for the null space is $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$. Since the pivots are in the 1st and 3rd columns, a basis for the column space is the 1st and 3rd columns of A, namely $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} \right\}$

2) Let $V = \mathbf{R}_1[t]$, the space of linear polynomials in a variable t. Let $\mathcal{E} = \{1, t\}$ and $\mathcal{B} = \{5 + 7t, 3 + 4t\}$ be two different bases for V. Let $\mathbf{v}(t) = 2 + t$, and let $L : V \to V$ be the linear transformation given by $L(\mathbf{p})(t) = \mathbf{p}(2t)$.

a) Compute the change-of-basis matrices $P_{\mathcal{E}\mathcal{B}}$ and $P_{\mathcal{B}\mathcal{E}}$.

Since $[\mathbf{b}_1]_{\mathcal{E}} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$ and $[\mathbf{b}_2]_{\mathcal{E}} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, we have $P_{\mathcal{E}\mathcal{B}} = \begin{pmatrix} 5 & 3 \\ 7 & 4 \end{pmatrix}$ and $P_{\mathcal{B}\mathcal{E}} = P_{\mathcal{E}\mathcal{B}}^{-1} = \begin{pmatrix} -4 & 3 \\ 7 & -5 \end{pmatrix}$.

b) Compute $[\mathbf{v}]_\mathcal{E}$ and $[\mathbf{v}]_\mathcal{B}$.

$[\mathbf{v}]_\mathcal{E} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $[\mathbf{v}]_\mathcal{B} = P_{\mathcal{B}\mathcal{E}}[\mathbf{v}]_\mathcal{E} = \begin{pmatrix} -4 & 3 \\ 7 & -5 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ 9 \end{pmatrix}$. You can check that $\mathbf{v}$ is in fact equal to $-5\mathbf{b}_1 + 9\mathbf{b}_2$.

c) Compute $[L]_\mathcal{E}$ and $[L]_\mathcal{B}$.

Since $L(1) = 1$ and $L(t) = 2t$, $[L]_\mathcal{E} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $[L]_\mathcal{B} = P_{\mathcal{B}\mathcal{E}}[L]_\mathcal{E} P_{\mathcal{E}\mathcal{B}} = \begin{pmatrix} 22 & 12 \\ -35 & -19 \end{pmatrix}$.

3. Let $V = M_{2,2}$ be the space of 2×2 real matrices, with basis $\mathcal{E} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$. Let $L : V \to V$ be a linear transformation given by $L(A) = A^T + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A$.

a) Compute $[L]_\mathcal{E}$.

We compute $L(e_1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^T + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = e_1 + e_3$. Similarly, $L(e_2) = e_3 + e_4$, $L(e_3) = e_1 + e_2$ and $L(e_4) = e_2 + e_4$, so

$$[L]_\mathcal{E} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

b) Find a basis for $Ker(L)$.

$[L]_\mathcal{E}$ row-reduces to $\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, whose null space is spanned by $\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$. This means that a basis for $Ker(L)$ is given by the vector whose coordinates are $(1, -1, -1, 1)^T$, namely $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$.

c) Find a basis for $Range(L)$.

There are pivots in the first three columns, so the first three columns of $[L]_{\mathcal{E}}$ form a basis for $Col([L]_{\mathcal{E}})$. Since $Range(L)$ consists of those vectors whose coordinates are in $Col([L]_{\mathcal{E}})$, a basis for $Ker(L)$ is $\{e_1 + e_3, e_3 + e_4, e_1 + e_2\} = \left\{ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\}$

4. a) Find the characteristic polynomial and the eigenvalues of the matrix $A = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$. You do **not** need to compute the eigenvectors.

$$p_A(\lambda) = \begin{vmatrix} \lambda - 3 & -2 \\ -1 & \lambda - 4 \end{vmatrix} = \lambda^2 - 7\lambda + 10.$$

The eigenvalues are the roots of $p_A(\lambda)$, namely 5 and 2.

b) The eigenvalues of the matrix $B = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix}$ are -2, 0, and 2. Find eigenvectors for each of these eigenvalues.

We get the eigenvectors by row-reducing $B + 2I$, B and $B - 2I$. The eigenvectors, in order, are $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$, and $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$.