M346 Third Midterm Exam Solutions, November 19, 2013

1) Consider the system of differential equations:

$$\begin{aligned} \frac{dx_1}{dt} &= x_1(4 - x_1 - 3x_2) \\ \frac{dx_2}{dt} &= x_2(1 + x_1 - 2x_2) \end{aligned}$$

[These come from a predator-prey system, with x_1 counting the prey and x_2 counting the predators.]

a) Find the fixed points. (There are four of them.)

Setting $dx_1/dt = 0$ implies that $x_1 = 0$ or $4 - x_1 - 3x_2 = 0$. Setting $dx_2/dt = 0$ implies that $x_2 = 0$ or $1 + x_1 - 2x_2 = 0$. This gives 4 possibilities, depending on which "or" we pick from each equation, leading to the fixed points $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1/2 \end{pmatrix}$, $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

b) For each fixed point, write down a system of **linear** differential equations that approximate the system near the fixed point.

We compute the matrix

$$A = \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{pmatrix} = \begin{pmatrix} 4 - 2x_1 - 3x_2 & -3x_1 \\ x_2 & 1 + x_1 - 4x_2 \end{pmatrix}$$

of partial derivatives. Evaluating this matrix at the four fixed points gives our linearized systems:

i) Near $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\mathbf{y} = \mathbf{x}$ and $\frac{d\mathbf{y}}{dt} \approx \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{y}$.

ii) Near $\begin{pmatrix} 0 \\ 1/2 \end{pmatrix}$, $\mathbf{y} = \mathbf{x} - \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}$ and $\frac{d\mathbf{y}}{dt} \approx \begin{pmatrix} 5/2 & 0 \\ 1/2 & -1 \end{pmatrix} \mathbf{y}$.

iii) Near $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$, $\mathbf{y} = \mathbf{x} - \begin{pmatrix} 4 \\ 0 \end{pmatrix}$ and $\frac{d\mathbf{y}}{dt} \approx \begin{pmatrix} -4 & -12 \\ 0 & 5 \end{pmatrix} \mathbf{y}$.

iv) Near $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\mathbf{y} = \mathbf{x} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\frac{d\mathbf{y}}{dt} \approx \begin{pmatrix} -1 & -3 \\ 1 & -2 \end{pmatrix} \mathbf{y}$.

c) For each fixed point, indicate how many stable, neutral and unstable modes there are, and whether the fixed point as a whole is stable, neutral or unstable.

We look at the eigenvalues of the matrix in each case:

i) 4 and 1 are positive, so both modes (and the system) are unstable.

ii) 5/2 is positive and -1 is negative, so there is one unstable mode and one stable mode. The system as a whole is unstable.

iii) 5 is positive and -4 is negative, so there is one unstable mode and one stable mode. The system as a whole is unstable.

iv) $(-3 \pm i\sqrt{11})/2$ both have negative real parts, so both modes (and the system) is stable. Over time, $\mathbf{x}$ will spiral into the fixed point $(1,1)^T$.

2. a) Let V be the subspace of $\mathbb{R}^5$ spanned by $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 9 \\ 16 \\ 25 \end{pmatrix} \right\}$. Find an orthogonal basis for V. (We are using the standard inner product.)

This is Gram-Schmidt on $\mathbb{R}^5$. Our basis is $\{\mathbf{y_1}, \mathbf{y_2}, \mathbf{y_3}\}$, where

$\mathbf{y_1} = \mathbf{x_1} = (1,1,1,1,1)^T$.

$\mathbf{y_2} = \mathbf{x_2} - \frac{\langle \mathbf{y_1} | \mathbf{x_2} \rangle}{\langle \mathbf{y_1} | \mathbf{y_1} \rangle} \mathbf{y_1} = (1,2,3,4,5)^T - 3(1,1,1,1,1)^T = (-2,-1,0,1,,2)^T$.

$$\begin{aligned} \mathbf{y_3} &= \mathbf{x_3} - \frac{\langle \mathbf{y_1} | \mathbf{x_3} \rangle}{\langle \mathbf{y_1} | \mathbf{y_1} \rangle} \mathbf{y_1} - \frac{\langle \mathbf{y_2} | \mathbf{x_3} \rangle}{\langle \mathbf{y_2} | \mathbf{y_2} \rangle} \mathbf{y_2} \\ &= (1,4,9,16,25)^T - 11(1,1,1,1,1)^T - 6(-2,-1,0,1,,2)^T \\ &= (2,-1,-2,-1,2)^T. \end{aligned}$$

b) Within $L^2([0,\pi])$, with inner product $\langle f | g \rangle = \int_0^\pi \overline{f(t)} g(t) dt$, let W be the span of $\sin(t)$ and $\sin^2(t)$. Find an orthogonal basis for W. You may use the following facts without explanation: $\int_0^\pi \sin^n(t) dt$ equals π if $n = 0$, 2 if $n = 1$, $\pi/2$ if $n = 2$ and $4/3$ if $n = 3$.

Our basis is $\{\mathbf{y_1}, \mathbf{y_2}\}$ where $\mathbf{y_1} = \mathbf{x_1} = \sin(t)$ and

$$\begin{aligned} \mathbf{y_2} &= \mathbf{x_2} - \frac{\langle \mathbf{y_1} | \mathbf{x_2} \rangle}{\langle \mathbf{y_1} | \mathbf{y_1} \rangle} \mathbf{y_1} \\ &= \sin^2(t) - \frac{4/3}{\pi/2} \sin(t) = \sin^2(t) - \frac{8}{3\pi} \sin(t). \end{aligned}$$

3. a) Find the best fit (least squares) line $y = c_0 + c_1 x$ through the points $(-2,-2)$, $(-1,2)$, $(0,2)$, $(1,4)$, and $(2,14)$.

$$A = \begin{pmatrix} 1 & -2 \\ 1 & -2 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} -2 \\ 2 \\ 2 \\ 4 \\ 14 \end{pmatrix}, \text{ so}$$

$$A^T A = \begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix}; \qquad A^T \mathbf{b} = \begin{pmatrix} 20 \\ 34 \end{pmatrix},$$

so $c_0 = 4$ $c_1 = 3.4$. Our best fit line is $y = 3.4x + 4$.

b) Find the best fit parabola $y = c_0 + c_1 t + c_2 t^2$ through the same points. (Note: Don't be surprised if you get different values of c_0 and c_1 than in part (a)).

$$\text{Now } A = \begin{pmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix}, \text{ so}$$

$$A^T A = \begin{pmatrix} 5 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 34 \end{pmatrix}; \qquad A^T \mathbf{b} = \begin{pmatrix} 20 \\ 34 \\ 54 \end{pmatrix},$$

Solving $A^T A \mathbf{c} = A^T \mathbf{b}$ (say, by row reduction) gives $c_0 = 2$, $c_1 = 3.4$ and $c_2 = 1$, so our best fit parabola is $y = 2 + 3.4x + x^2$.

4. Consider the Hermitian matrix $H = \begin{pmatrix} 4 & 4i \\ -4i & -2 \end{pmatrix}$

a) Find the eigenvalues and eigenvectors of H.

The eigenvalues are 6 and -4, with eigenvectors $\begin{pmatrix} 2i \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2i \end{pmatrix}$, respectively.

b) Construct an orthonormal basis of $\mathbb{C}^2$ consisting of eigenvectors of H.

Since our eigenvectors are already orthogonal, we just have to normalize them: Our basis is $\left\{ \frac{1}{\sqrt{5}} \begin{pmatrix} 2i \\ 1 \end{pmatrix}, \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2i \end{pmatrix} \right\}$.

c) Construct (explicitly!) another matrix T with eigenvalues i and $-i$, whose eigenvectors are the same as those of H. What sort of matrix is T? What can you say about the columns of T?

$T = PDP^{-1} = \frac{\pm 1}{5}\begin{pmatrix} -3i & 4 \\ -4 & 3i \end{pmatrix}$, where the $\pm$ depends on which eigenvector of H you chose to have eigenvalue i and which to have eigenvalue $-i$. This calculation is made easier by the fact that P is a unitary matrix, so $P^{-1} = P^\dagger$. Since the eigenvalues of T are on the unit circle and the eigenvectors are orthogonal, T must be unitary, and you can check that the columns are orthonormal.