M346 Second Midterm Exam, October 22, 2013

1) Find the eigenvalues of the matrix

$$A = \begin{pmatrix} 5 & 3 & 0 & 0 \\ -3 & -1 & 0 & 3 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 4 & 3 \end{pmatrix}.$$

For each eigenvalue, indicate what the geometric and algebraic multiplicity is. If the matrix is diagonalizable, find a basis of $\mathbb{R}^4$ consisting of eigenvectors of A. If the matrix is not diagonalizable, do the next best thing and find a basis consisting of power vectors.

2. Do ONE and ONLY one of the following two problems. [If you attempt both, cross out the one you don't want graded. If it's not clear which you intend, I'll grade (a) and ignore (b).]

a) Compute e^{At}, where $A = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}$.

b) Or compute B^n, where $B = \begin{pmatrix} 3 & 0 \\ 5 & 3 \end{pmatrix}$

3. The matrix $A = \begin{pmatrix} -2 & 3 \\ 2 & -1 \end{pmatrix}$ has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -4$ and corresponding eigenvectors $\mathbf{b_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{b_2} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$.

a) Solve the system of difference equations

$$\begin{aligned} x_1(n) &= -2x_1(n-1) + 3x_2(n-1) \\ x_2(n) &= 2x_1(n-1) - x_2(n-1) \end{aligned}$$

with initial conditions $\mathbf{x}(0) = \begin{pmatrix} -7 \\ 8 \end{pmatrix}$.

b) Solve the system of differential equations

$$\begin{aligned} \frac{dx_1}{dt} &= -2x_1 + 3x_2 \\ \frac{dx_2}{dt} &= 2x_1 - x_2 \end{aligned}$$

with initial conditions $\mathbf{x}(0) = \begin{pmatrix} -7 \\ 8 \end{pmatrix}$.

c) Solve the system of second order differential equations

$$\begin{aligned} \frac{d^2x_1}{dt^2} &= -2x_1 + 3x_2 \\ \frac{d^2x_2}{dt^2} &= 2x_1 - x_2 \end{aligned}$$

with initial conditions $\mathbf{x}(0) = \begin{pmatrix} -7 \\ 8 \end{pmatrix}$ and $\dot{\mathbf{x}}(0) = \begin{pmatrix} -5 \\ 5 \end{pmatrix}$.

4. A matrix A has eigenvalues -4 and $1/2$ and eigenvectors $\mathbf{b}_1 = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$ and $\mathbf{b}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$. Suppose that $\mathbf{x}(0) = c_1\mathbf{b}_1 + c_2\mathbf{b}_2$, where c_1 and c_2 are both nonzero.

a) Describe the long-time behavior of the solution to $\mathbf{x}(n) = A\mathbf{x}(n-1)$. Which mode(s) grow(s) and which shrink? In what direction will $\mathbf{x}(n)$ be pointing in the long run, and how fast will it be growing or shrinking?

b) Describe the long-time behavior of the solution to $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$. Which mode(s) grow(s) and which shrink? In what direction will $\mathbf{x}(t)$ be pointing in the long run, and how fast will it be growing or shrinking?