M346 Third Midterm Exam, November 19, 2013

(2 pages) 1) Consider the system of differential equations:

$$\begin{aligned}\frac{dx_1}{dt} &= x_1(4 - x_1 - 3x_2)\\ \frac{dx_2}{dt} &= x_2(1 + x_1 - 2x_2)\end{aligned}$$

[These come from a predator-prey system, with x_1 counting the prey and x_2 counting the predators.]

a) Find the fixed points. (There are four of them.)

b) For each fixed point, write down a system of **linear** differential equations that approximate the system near the fixed point.

c) For each fixed point, indicate how many stable, neutral and unstable modes there are, and whether the fixed point as a whole is stable, neutral or unstable.

2. a) Let V be the subspace of $\mathbb{R}^5$ spanned by $\left\{\begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\3\\4\\5\end{pmatrix}, \begin{pmatrix}1\\4\\9\\16\\25\end{pmatrix}\right\}$. Find an orthogonal basis for V. (We are using the standard inner product.)

b) Within $L^2([0,\pi])$, with inner product $\langle f|g\rangle = \int_0^\pi \overline{f(t)}g(t)dt$, let W be the span of $\sin(t)$ and $\sin^2(t)$. Find an orthogonal basis for W. You may use the following facts without explanation: $\int_0^\pi \sin^n(t)dt$ equals π if $n = 0$, 2 if $n = 1$, $\pi/2$ if $n = 2$ and $4/3$ if $n = 3$.

3. a) Find the best fit (least squares) line $y = c_0 + c_1 x$ through the points $(-2,-2)$, $(-1,2)$, $(0,2)$, $(1,4)$, and $(2,14)$.

b) Find the best fit parabola $y = c_0 + c_1 t + c_2 t^2$ through the same points. (Note: Don't be surprised if you get different values of c_0 and c_1 than in part (a)).

4. Consider the Hermitian matrix $H = \begin{pmatrix} 4 & 4i \\ -4i & -2 \end{pmatrix}$

a) Find the eigenvalues and eigenvectors of H.

b) Construct an orthonormal basis of $\mathbb{C}^2$ consisting of eigenvectors of H.

c) Construct (explicitly!) another matrix T with eigenvalues i and $-i$, whose eigenvectors are the same as those of H. What sort of matrix is T? What can you say about the columns of T?