

M346 Final Exam, May 19, 2009

1. For each of these collections \mathcal{B} of vectors in a vector space V , indicate (with explanation) whether \mathcal{B} is linearly independent, spans V , both, or neither.

a) In R^3 , $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 10 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} \right\}.$

This boils down to row-reducing the matrix $A = \begin{pmatrix} 1 & 3 & 2 & 1 \\ 2 & 6 & 4 & 3 \\ 3 & 10 & 5 & 4 \end{pmatrix}$ to $\begin{pmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. The rank is 3, so the vectors span R^3 but are linearly dependent.

b) In R^4 , $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 10 \\ 5 \\ 4 \end{pmatrix} \right\}.$

This boils down to row-reducing A^T . The rank is still 3, so the vectors are linearly independent but do not span.

c) In $R_2[t]$, $\mathcal{B} = \{1 + 2t + 3t^2, 3 + 6t + 10t^2, 1 + 3t + 4t^2\}.$

Working in the standard basis, this boils down to row-reducing $\begin{pmatrix} 1 & 3 & 1 \\ 2 & 6 & 3 \\ 3 & 10 & 4 \end{pmatrix}$, in other words A with the third column removed. The result is the identity matrix, showing that \mathcal{B} is linearly independent, spans, and hence is a basis for $R_2[t]$.

d) In $R_3[t]$, $\mathcal{B} = \{1 + 2t + 3t^2, 3 + 6t + 10t^2, 1 + 3t + 4t^2\}.$

These are the same vectors as in part (c), so they are still linearly independent, but now they are in a bigger space and don't span.

2. In $R_2[t]$, consider the bases $\mathcal{E} = \{1, t, t^2\}$ and $\mathcal{B} = \{2, 2t + 5, t^2 + 5t + 7\}$, and the linear transformation $L : R_2[t] \rightarrow R_2[t]$, $L\mathbf{p}(t) = \mathbf{p}(t) + \mathbf{p}'(t)$.

a) Find the change-of-basis matrices $P_{\mathcal{B}\mathcal{E}}$ and $P_{\mathcal{E}\mathcal{B}}$.

$$P_{\mathcal{E}\mathcal{B}} = \begin{pmatrix} 2 & 5 & 7 \\ 0 & 2 & 5 \\ 0 & 0 & 1 \end{pmatrix}. \quad \text{By row-reduction, we compute } P_{\mathcal{B}\mathcal{E}} = P_{\mathcal{E}\mathcal{B}}^{-1} =$$

$$\begin{pmatrix} 1/2 & -5/4 & 11/4 \\ 0 & 1/2 & -5/2 \\ 0 & 0 & 1 \end{pmatrix}.$$

b) If $\mathbf{p}(t) = t^2 + 9t + 23$, find $[\mathbf{p}]_{\mathcal{B}}$.

$$[\mathbf{p}]_{\mathcal{B}} = P_{\mathcal{B}\mathcal{E}}[\mathbf{p}]_{\mathcal{E}} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}. \text{ You should check that } \mathbf{p}(t) \text{ really is}$$

$$3(2) + 2(2t + 5) + (t^2 + 5t + 7).$$

c) Find $[L]_{\mathcal{E}}$ and $[L]_{\mathcal{B}}$.

$$[L]_{\mathcal{E}} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } [L]_{\mathcal{B}} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}. \text{ The latter can be obtained}$$

either from $P_{\mathcal{B}\mathcal{E}}[L]_{\mathcal{E}}P_{\mathcal{E}\mathcal{B}}$ or from the fact that each basis element is the derivative of the one after it, so $L - I$ is just a shift downwards.

3. (a) Find a 2×2 matrix A whose eigenvalues are -30 and 40 and whose corresponding eigenvectors are $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$. [Hint: the final answer should only involve integers, although you may see some fractions along the way.]

$$P = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}, D = \begin{pmatrix} -30 & 0 \\ 0 & 40 \end{pmatrix}, \text{ and } A = PDP^{-1} = \begin{pmatrix} -44 & 56 \\ -21 & 54 \end{pmatrix}.$$

(b) What are the eigenvalues of $A^2 - 10A$?

Since the eigenvalues of A are -30 and 40 , the eigenvalues of $A^2 - 10A$ must be $(-30)^2 - 10(-30) = 1200$ and $40^2 - 10(40) = 1200$.

c) Compute $A^2 - 10A$. (No, you do NOT need a calculator to do this.)

Since $A^2 - 10A$ is diagonalizable with both eigenvalues equal to 1200 , $A^2 - 10A = \begin{pmatrix} 1200 & 0 \\ 0 & 1200 \end{pmatrix}$. You *could* get this result by multiplying out A^2 and subtracting $10A$, but that would be very, very painful.

4. a) Find the eigenvalues and eigenvectors of $A = \begin{pmatrix} 9 & -7 \\ 4 & -2 \end{pmatrix}$.

Since the trace is 7 and the determinant is 10 , the eigenvalues are 2 and

5. By row-reduction, the eigenvectors are $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 7 \\ 4 \end{pmatrix}$.

b) Compute $e^{i\pi A}$. [The final answer involves rational numbers with small denominators.]

$e^{2\pi i} = 1$ and $e^{5\pi i} = -1$, so we are looking for a matrix with eigenvectors

$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 7 \\ 4 \end{pmatrix}$ and eigenvalues 1 and -1 . This is $\begin{pmatrix} 1 & 7 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 7 \\ 1 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} -11/3 & 14/3 \\ -8/3 & 11/3 \end{pmatrix}$.

5. Consider the system of differential equations

$$\begin{aligned} \frac{dx_1}{dt} &= x_1 \\ \frac{dx_2}{dt} &= 2x_1 + x_2 + 2x_3 \\ \frac{dx_3}{dt} &= 3x_1 + 2x_2 + x_3 \end{aligned}$$

a) Find the general solution.

Our matrix is $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix}$ with eigenvalues 1, 3 and -1 and eigenvectors $(2, -3, -2)^T$, $(0, 1, 1)^T$ and $(0, 1, -1)^T$. The general solution is then

$$\mathbf{x}(t) = c_1 e^t \begin{pmatrix} 2 \\ -3 \\ -2 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_3 e^{-t} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix},$$

where c_1 , c_2 and c_3 are arbitrary constants.

b) If $\mathbf{x}(0) = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$, what is the limiting value of $\frac{x_1(t)}{x_2(t)}$ as $t \rightarrow \infty$?

The dominant eigenvector is $\lambda_2 = 3$, and a quick calculation shows that $c_2 \neq 0$. This means that $\mathbf{x}(t)$ asymptotically points in the \mathbf{b}_2 direction, with x_1/x_2 approaching $0/1 = 0$.

6. Use the Gram Schmidt process to convert the following basis for a 3-dimensional subspace of R^4 into an orthonormal basis for that subspace.

$$\mathbf{x}_1 = (1, 1, -1, 0)^T, \mathbf{x}_2 = (4, 5, 0, 4)^T, \mathbf{x}_3 = (-2, 3, -2, -7)^T.$$

$$|\mathbf{y}_1\rangle = |\mathbf{x}_1\rangle = (1, 1, -1, 0)^T.$$

$$|\mathbf{y}_2\rangle = |\mathbf{x}_2\rangle - \frac{\langle \mathbf{y}_1 | \mathbf{x}_2 \rangle}{\langle \mathbf{y}_1 | \mathbf{y}_1 \rangle} |\mathbf{y}_1\rangle = \mathbf{x}_2 - 3\mathbf{y}_1 = (1, 2, 3, 4)^T.$$

$$|\mathbf{y}_3\rangle = |\mathbf{x}_3\rangle - \frac{\langle \mathbf{y}_1 | \mathbf{x}_3 \rangle}{\langle \mathbf{y}_1 | \mathbf{y}_1 \rangle} |\mathbf{y}_1\rangle - \frac{\langle \mathbf{y}_2 | \mathbf{x}_3 \rangle}{\langle \mathbf{y}_2 | \mathbf{y}_2 \rangle} |\mathbf{y}_2\rangle = \mathbf{x}_3 - \mathbf{y}_1 + \mathbf{y}_2 = (-2, 4, 2, -3)^T.$$

Normalizing, we get $\mathbf{z}_1 = (1, 1, -1, 0)^T/\sqrt{3}$, $\mathbf{z}_2 = (1, 2, 3, 4)^T/\sqrt{30}$, and $\mathbf{z}_3 = (-2, 4, 2, -3)^T/\sqrt{33}$.

7. Find all least-squares solutions to the system of equations

$$\begin{aligned}x_1 + 2x_2 &= -3 \\3x_1 + 2x_2 &= 7 \\2x_1 - 2x_2 &= 14 \\4x_1 - x_2 &= 5\end{aligned}$$

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \\ 2 & -2 \\ 4 & -1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} -3 \\ 7 \\ 14 \\ 5 \end{pmatrix}, A^T A = \begin{pmatrix} 30 & 0 \\ 0 & 13 \end{pmatrix} \text{ and } A^T \mathbf{b} = \begin{pmatrix} 66 \\ -25 \end{pmatrix},$$

so the unique solution is $x_1 = 66/30 = 11/5$ and $x_2 = -25/13$. [This is the only place in the test where the denominators exceed 10.]

8. Consider a wave $f(x, t)$ on the interval $[0, 1]$, with Dirichlet boundary conditions ($f(0, t) = f(1, t) = 0$ for all time), moving with velocity 1. The initial condition is $f(x, 0) = \sin(\pi x) + \sin(2\pi x)$ and $\frac{\partial f}{\partial t}(x, 0) = 0$.

a) Find $f(x, t)$ for all x and all t .

This problem can be done either with traveling waves or with standing waves. Standing waves are much easier, but I'll show both ways of doing it.

In terms of standing waves, $f(x, t) = \sum_n \sin(n\pi x)[a_n \cos(n\pi t) + b_n \sin(n\pi t)]$. From the initial conditions we get $a_1 = a_2 = 1$, all other a 's are zero, and all b 's are zero. In other words, $f(x, t) = \sin(\pi x) \cos(\pi t) + \sin(2\pi x) \cos(2\pi t)$. That's the sum of two standing waves with different frequency.

In terms of traveling waves, we get $h_1(x) = h_2(x) = f(x, 0)/2$, so $f(x, t) = \frac{1}{2}[\sin(\pi(x-t)) + \sin(2\pi(x-t)) + \sin(\pi(x+t)) + \sin(2\pi(x+t))]$. Several people got this right, but *nobody* was able to take this result and solve part (b) with it.

b) Sketch $f(x, t)$ at times $t = 1/4$, $t = 1/2$, $t = 1$, $t = 3/2$, and $t = 2$. I've sketched $f(0, t)$ on the board to get you started.

At $t = 1/4$, $\cos(\pi t) = \sqrt{2}/2$ and $\cos(2\pi t) = 0$, so $f(x, 1/4) = \sqrt{2} \sin(\pi x)/2$.

At $t = 1/2$, $\cos(\pi t) = 0$ and $\cos(2\pi t) = -1$, so $f(x, 1/2) = -\sin(2\pi x)$.

At $t = 1$, $\cos(\pi t) = -1$ and $\cos(2\pi t) = 1$, so $f(x, 1) = -\sin(\pi x) + \sin(2\pi x)$.

At $t = 3/2$, $\cos(\pi t) = 0$ and $\cos(2\pi t) = -1$, so $f(x, 3/2) = -\sin(2\pi x)$, just as at $t = 1/2$.

At $t = 2$, $\cos(\pi t) = 1$ and $\cos(2\pi t) = 1$, so $f(x, 2) = \sin(\pi x) + \sin(2\pi x)$.

I can't draw the pictures online, but the picture at $t = 1$ looks just like the picture at $t = 0$, only rotated by 180 degrees, with a small positive bump on the left and a large negative bump on the right. At $t = 2$ the picture is just like at $t = 0$, with a large positive bump on the left and a small negative bump on the right.

You can also get these results using traveling waves, that is by taking $f(x, 0)$, extending it to an odd periodic function, shifting it a distance t to the left and t to the right, adding the shifted waves, and dividing by 2. It works, but it's a lot harder that way.