M408R Final Exam, December 12, 2014

1) Plug-and-chug grab bag (2 pages): Each part is worth 4 points.

a) Compute the derivative of $x^3 \ln(x)$.

The product rule gives $x^3/x + 3x^2 \ln(x) = 3x^2 \ln(x) + x^2$.

b) Compute the derivative of $\dfrac{xe^x}{\sin(x)}$

This is a combination of the quotient rule and the product rule. Applying the product rule to $(xe^x)'$ gives $(x+1)e^x$. Then the quotient rule gives

$$\frac{d}{dx}\frac{xe^x}{\sin(x)} = \frac{\sin(x)(x+1)e^x - xe^x\cos(x)}{\sin^2(x)}$$

c) Compute the derivative of x^e.

This is just a power: $(x^e)' = ex^{e-1}$.

d) Find all solutions to $10^{(x^2)} = 1,000,000,000$.

Since $10^{x^2} = 10^9$, we must have $x^2 = 9$, so $x = \pm 3$.

e) Simplify: $2\log_3(45) - \log_3(25)$.

This is $2\log_3(45) - 2\log_3(5) = 2\log_3(45/5) = 2\log_3(9) = 4$.

f) Evaluate: $\int_1^2 6x^2 - 8x + 2dx$.

This is $2x^3 - 4x^2 + 2x\Big|_1^2 = (16 - 16 + 4) - (2 - 4 + 2) = 4 - 0 = 4$.

g) Evaluate: $\int (3x^2 + 7)\ln(x)dx$.

Integrate by parts with $u = \ln(x)$, $du = dx/x$, $dv = (3x^2+7)dx$, $v = x^3 + 7x$, so our answer is

$$uv - \int vdu = (x^3+7x)\ln(x) - \int x^2 + 7dx = (x^3+7x)\ln(x) - (x^3/3 + 7x) + C.$$

h) Evaluate: $\int_0^1 6x\sqrt{1-x^2}dx$.

This is a u-substitution with $u = 1 - x^2$ and $du = -2xdx$. Note that $u = 1$ when $x = 0$ and $u = 0$ when $x = 1$, so our integral turns into

$$\int_1^0 -3\sqrt{u}du = -2u^{3/2}\Big|_1^0 = 0 - (-2) = 2.$$

i) Evaluate: $\dfrac{d}{dx}\displaystyle\int_3^x \frac{\ln(t+5)}{1+\tan^{-1}(3t)}dt$

By the (first) fundamental theorem of calculus, this is $\frac{\ln(x+5)}{1+\tan^{-1}(3x)}$.

j) Evaluate: $\displaystyle\int \frac{xe^x dx}{(x+1)^2}$. [Hint: Integration by parts with $u = xe^x$ works. More obvious choices don't.]

$u = xe^x$, $du = (x+1)e^x dx$ (by the product rule), $dv = (x+1)^{-2}dx$, and $v = -(x+1)^{-1}$. We then have

$$uv - \int v du = \frac{-xe^x}{x+1} + \int e^x dx = e^x(1 - \frac{x}{x+1}) + C = \frac{e^x}{x+1} + C.$$

2) Derivatives. Suppose that $f(x)$ and $g(x)$ are differentiable functions, that $g(1) = 3$, $g'(1) = 2$, $f(3) = 5$, and $f'(3) = -1/4$. Let $h(x) = f(g(x))$.

a) Using the microscope equation for g, find the approximate value of $g(1.06)$. Call this number a.

$$g(1.06) \approx g(1) + 0.06g'(1) = 3 + 0.12 = 3.12 = a$$

b) Using the microscope equation for f, find the approximate value of $f(a)$.

$$f(3.12) \approx f(3) + 0.12f'(3) = 5 - 0.03 = 4.97.$$

c) Find $h'(1)$.

By the chain rule, $h'(1) = f'(g(1))g'(1) = f'(3)g'(1) = -2/4 = -1/2$.

d) Use the microscope equation for h to find the approximate value of $h(1.06)$.

$$h(1.06) \approx h(1) + 0.06h'(1) = f(g(1)) - 0.03 = 5 - 0.03 = 4.97.$$

3) A Thanksgiving turkey comes out of the oven when the meat is 170° (Fahrenheit). It cools at a rate proportional to the difference between the turkey's temperature and room temperature (which we'll take to be 70°).

a) Write down a *differential (aka rate) equation* that describes this cooling process. Be sure to explain what each of your variables means, and what each of your parameters mean.

If t is time (in minutes) and $T(t)$ is the temperature in degrees Fahrenheit at time t, then we are told that the rate of cooling ($-T'$) is a constant times $T - 70$. That is,

$$T' = -r(T - 70),$$

where r is an unknown parameter that describes how fast the turkey cools.

b) Now solve the differential equation. What is the temperature of the turkey as a function of time? (This answer will still involve parameter(s).)

The solution to this sort of differential equation is $T(t) = 70 + Ae^{-rt}$. Since $T(0) = 170$, the constant A must equal 100.

c) Suppose that after 10 minutes the turkey has cooled to 140°. Using that information, find the values of the parameter(s) in your differential equation. This should give you the temperature as an explicit function of time.

Since $T(10) = 140$, we must have $100e^{-10r} = 140 - 70 = 70$, so $-10r = \ln(0.7)$, and $r = -\ln(0.7)/10 \approx .03567$

d) Following up on part (c), after how much time will the turkey be 100° (and ready to carve and eat)?

We need to solve $100 = 70+100e^{-rt}$, so $e^{-rt} = 30/100$, so $t = -\ln(0.3)/r \approx 33.76$. In other words, the turkey will be ready to carve 33.76 minutes after it comes out of the oven.

4) (2 pages) Charged particles in magnetic fields tend to move along helices centered on the field lines. This is what causes the Aurora Borealis. The following problem explores the math behind behind this phenomenon.

Let $v_x(t)$, $v_y(t)$ and $v_z(t)$ be the components of an electron's velocity in the x, y and z directions at time t. If the magnetic field points in the z direction, then the laws of physics say these variables will satisfy the following rate equations:

$$\begin{aligned} v_x' &= cv_y \\ v_y' &= -cv_x \end{aligned}$$

$$v_z' = 0,$$

where c is a parameter that depends on the strength of the magnet.

a) Suppose that $c = 1/2$, that $v_x(0) = 10$, $v_y(0) = 20$, and $v_z(0) = 15$. Use Euler's method, with step size $h = 0.2$, to estimate $v_x(0.4)$, $v_y(0.4)$, and $v_z(0.4)$. (Along the way you'll also compute $v_x(0.2)$, $v_y(0.2)$ and $v_z(0.2)$.)

We make a table:

t	v_x	v_y	v_z	v_x'	v_y'	v_z'
0	10	20	15	10	-5	0
0.2	12	19	15	9.5	-6	0
0.4	13.9	17.8	15			

On each row, the entries for v_x', v_y' and v_z' are computed from the values of v_x, v_y, v_z and the rate equation. The values for v_x, v_y and v_z are computed from the previous row and the microscope equation: $f(a+h) \approx f(a)+hf'(a)$.

b) The exact solution to the initial value problem is:

$$\begin{aligned} v_x(t) &= 10\cos(t/2) + 20\sin(t/2), \\ v_y(t) &= 20\cos(t/2) - 10\sin(t/2), \\ v_z(t) &= 15. \end{aligned}$$

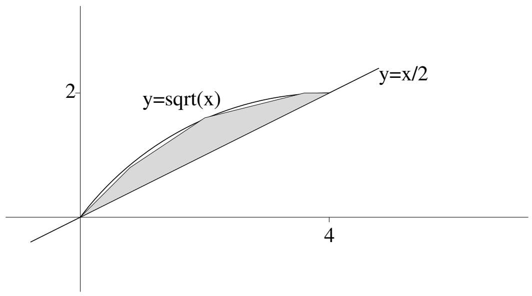
Now that we know the components of the velocity, we can infer the coordinates $x(t)$, $y(t)$ and $z(t)$ of the position. Assuming that $x(0) = y(0) = z(0) = 0$, find $x(t)$, $y(t)$, and $z(t)$ (exactly).

$x(t)$ is an anti-derivative of $v_x(t)$, and so takes the form $x(t) = 20\sin(t/2) - 40\cos(t/2) + C$. Since $x(0) = 0$ and $\cos(0) = 1$, we must have $C = 40$. Similarly, $y(t) = 40\sin(t/2) + 20\cos(t/2) + \tilde{C}$, and since $y(0) = 0$, we must have $\tilde{C} = -20$. Finally, $z(t) = 15t + C'$, and $C' = 0$ since $z(0) = 0$. So the bottom line is:

$$\begin{aligned} x(t) &= 20\sin(t/2) - 40\cos(t/2) + 40 \\ y(t) &= 40\sin(t/2) + 20\cos(t/2) - 20 \\ z(t) &= 15t \end{aligned}$$

This trajectory follows a helix centered on the line $x = 40$, $y = -20$.

5) Vertical and horizontal slices. We wish to determine the area of the shaded region in the figure below, between the curve $y = \sqrt{x}$ and $y = x/2$.



a) Express this area as an integral over x. (That is, slice it into vertical slices). Be clear about what function(s) you are integrating, and about the limits of integration. Do not evaluate the integral (yet).

Slicing vertically, we have that each slice is a rectangle of height $\sqrt{x} - (x/2)$, since the top of the slice is at $y = \sqrt{x}$ and the bottom is at $y = x/2$. The area is then $(\sqrt{x} - x/2)\Delta x$. Adding things up and taking a limit gives

$$\int_{x=0}^{4} \sqrt{x} - \frac{x}{2} dx.$$

b) Express this area as an integral over y. (That is, slice it into horizontal slices). Be clear about what function(s) you are integrating, and about the limits of integration. Do not evaluate the integral (yet).

Slicing horizontally, we have that each slice is a rectangle of width $2y - y^2$, since the right end of the slice is at $y = x/2$, hence $x = 2y$, and the left end is at $y = \sqrt{x}$, hence $x = y^2$. The area is then $(2y - y^2)\Delta y$. Adding things up and taking a limit gives

$$\int_{y=0}^{2} (2y - y^2) dy.$$

c) Evaluate one of these two integrals (your choice) to get the area. Your final answer should be a number, like 3 or 17 or $5\sqrt{2}$.

The first integral gives $\frac{2}{3}x^{3/2} - \frac{x^2}{4}\Big|_0^4 = \frac{16}{3} - 4 = \frac{4}{3}$. The second integral gives
$y^2 - \frac{y^3}{3}\Big|_0^2 = 4 - \frac{8}{3} = \frac{4}{3}$. No matter how you slice it, the answer is $4/3$.

6) Growth spurts. According to a certain model (that I just made up), teenage boys grow at an average rate of $12te^{-t}$ inches/year, where t is the age in years minus 13. (That is $t = 3$ means "16 years old".)

a) Write down an integral that gives the average height gain between a boy's 14th and 18th birthdays. (You do not need to evaluate the integral.)

Let $r(t) = 12te^{-t}$. The growth in a time period Δt is then $r(t)\Delta t$. Adding things up and taking a limit gives an integral

$$\int_1^5 r(t)dt = \int_1^5 12te^{-t}dt.$$

b) If a boy is 4 foot 10 inches high on his 13th birthday, how tall can he expect to get by the time he is 17? Note that this part uses different start and end times than part (a). Your answer should be a precise number, like "5 foot 4.5 inches" (not the correct answer, BTW).

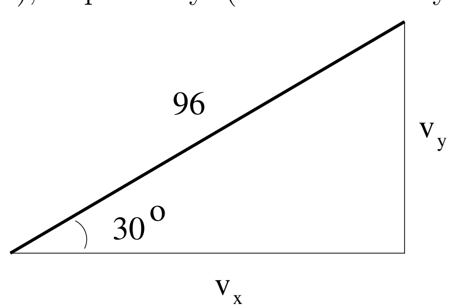
The growth from age 13 to age 17 is $\int_0^4 12te^{-t}dt$. This is done by integration by parts, with $u = t$. The result is

$$-12(t+1)e^{-t}\Big|_0^4 = 12 - 60e^{-4} \approx 10.9 \text{ inches.}$$

Adding this to his original height, he can expect to be around 5 foot 8.9 inches. (He'll eventually top out around 5 foot 10.)

7) (2 pages) This problem walks you through the following physics problem: "A quarterback throws a pass at 96ft/sec aimed 30° above horizontal. Ignoring air resistance, how far will the ball fly before being caught (at the same height as it was thrown)?"

a) If a ball is moving with speed 96 feet/sec in a direction 30° above horizontal, what are the x and y components of its velocity? Call these $v_x(0)$ and $v_y(0)$, respectively. (This isn't really a calculus problem. It's a trig problem.)



$v_x(0) = 96\cos(30°) = 48\sqrt{3} \approx 83.14$ feet/second. $v_y(0) = 96\sin(30°) = 48$ feet/second.

b) For now, ignore the horizontal part of the motion. We're just going to study the up-and-down motion. If the vertical acceleration is $a_y(t) = -32$ feet/sec^2 and the initial vertical velocity is given by your answer to part (a), what is the vertical velocity $v_y(t)$ for all time?

Since $v_y(t)$ is an anti-derivative of -32 and starts at 48, it must be $v_y(t) = 48 - 32t$ (feet/second).

c) If the vertical velocity is given by your answer to (b) and the initial height is $y(0) = 6$ (the height of the quarterback), what is $y(t)$ for all time?

Again we take an antiderivative to get $y(t) = 48t - 16t^2 + C$. Since $y(0) = 6$, $C = 6$, so $y(t) = 48t - 16t^2 + 6$.

d) At what positive time t is $y(t)$ again equal to 6 (the height of the receiver)? (That's the time at which the ball is caught.)

Solving $y(t) = 6$ gives $48t - 16t^2 = 0$. This has two solutions, namely the launch time $t = 0$ and the later time $t = 3$.

e) In the mean time, the horizontal velocity has not changed at all. If $x(0) = 0$, what is x when the ball is caught?

Since v_x is a constant, the ball travels a horizontal distance of $3v_x = 83.14$ yards (249.42 feet) in 3 seconds. If the ball was launched from the QB's own 20 yard line, it was caught 3 yards into the end zone. Touchdown!