M408N First Midterm Exam Solutions, September 24, 2014

1) Let $Q(t)$ be the percentage of students at UT who have dropped a class in week t, and suppose that the rate equation for Q is

$$Q' = 0.2Q - 0.005Q^2$$

and that $Q(5) = 10$.

a) Use Euler's method with step size $h = 2$ to estimate $Q(7)$.

One step of Euler's method is really just the microscope equation: $Q'(5) = 0.2(10) - 0.005(10)^2 = 1.5$, so $Q(7) \approx Q(5) + 2Q'(5) = 10 + 3 = 13$.

b) Use Euler's method with step size $h = 2$ to estimate $Q(3)$.

$Q(3) \approx Q(5) - 2Q'(5) = 10 - 3 = 7$.

c) Use Euler's method with step size $h = 1$ to estimate $Q(7)$.

$Q(6) \approx Q(5) + 1Q'(5) = 10 + 1.5 = 11.5$. We then compute $Q'(6) \approx 0.2(11.5) - 0.005(11.5)^2 = 1.63875$ and $Q(7) \approx Q(6) + 1Q'(6) \approx 11.5 + 1.63875 = 13.13875$.

2) Here is a table of values of the function $f(x) = \tan(x \text{ degrees})$.

x (in degrees)	$tan(x)$
44	0.9656887748
44.9	0.99651541969
44.99	0.99965099505
45	1
45.01	1.00034912679
45.1	1.00349676506
46	1.03553031379

a) Find $f'(45)$ to at least 4 decimal places. [Note: you may have already learned a formula for the derivative of the tangent function, but that formula probably uses radians rather than degrees, so it gives the wrong answer.]

If you use forward differences, the best estimate is

$$f'(45) \approx \frac{f(45.01) - f(45)}{.01} = \frac{0.00034912679}{0.01} = 0.034912679.$$

Using backwards differences, we get

$$f'(45) \approx \frac{f(45) - f(44.99)}{0.01} = \frac{0.000349004941}{0.01} = 0.0349004941.$$

Using centered differences gives

$$f'(45) \approx \frac{f(45.01) - f(44.99)}{0.02} = \frac{.0006981317290}{0.02} = 0.03490658645.$$

No matter how you do it, the answer to 4 decimal places is 0.0349.

b) Use this information and the microscope equation to estimate $f(50)$.

By the microscope equation $f(50) \approx f(45) + 5f'(45) = 1 + 5(0.0349) = 1.1745$. [FWIW, the actual value of $f(50)$ turns out to be 1.19175]

3) Suppose that $f(x)$ is a differentiable function with $f(2) = -3$ and $f'(2) = 7$.

a) Find an equation for the line tangent to $y = f(x)$ at $(2, -3)$.

Since the slope is 7, point-slope form says that $y + 3 = 7(x - 2)$, or $y = 7(x - 2) - 3$.

b) Estimate the values of $f(2.05)$ and $f(1.9)$.

$f(2.05) \approx -3 + 7(.05) = -2.65$. $f(1.9) \approx -3 + 7(-0.1) = -3.7$.

4) The following model is NOT the SIR model, but it uses the same sort of reasoning as the SIR model. It's up to you to provide the details.

A hospital is treating patients who have a particular non-fatal disease. On average, patients spend 8 days in the hospital before being released. Let $P(t)$ be the number of patients in the hospital at time t, and let $R(t)$ be the number of patients who have been released. Every day, 20 new patients are admitted to the hospital.

a) Write down a set of rate equations for P and R. **Explain your reasoning!!** What does each term in the rate equation represent?

Since the disease runs for 8 days (on average), we expect 1/8 of the patients to recover on any given day. So $R' = (1/8)P$. Meanwhile, P is increasing by 20 new patients and decreasing by $P/8$, for a total of:

$$P' = 20 - (P/8); \qquad R' = P/8.$$

b) If there are 100 patients in the hospital at a particular time, is the number of patients increasing or decreasing? What if there are 200 patients?

When $P = 100$, $P' = 20 - 100/8 = 7.5 > 0$, so the number of patients is increasing. When $P = 200$, $P' = 20 - 200/8 = -5$, so P is decreasing.

5) a) Find the derivative of the function $f(t) = t^3 - 3t + 2$. You may use the formulas from Section 3.5.

The derivative of t^3 is $3t^2$, the derivative of $-3t$ is -3, and the derivative of 2 is 0, so the derivative of $t^3 - 3t + 2$ is $3t^2 - 3 = 3(t^2 - 1)$.

b) The position of a particle is given by $x(t) = t^3 - 3t + 2$, where t is measured in seconds and x is measured in feet. How fast is the particle moving at time $t = 2$? Is it moving forwards or backwards?

When $t = 2$, $x' = 3(2)^2 - 3 = 9$, so the particle is moving forwards at a rate of 9 feet/second.

c) At what time(s) is the particle's velocity equal to zero?

Setting $x' = 0$ gives $t^2 - 1 = 0$, hence $t = \pm 1$.