Math 1310: CSM **Tutorial: Population Growth**

For this tutorial, you will need the Sage program Unlimited.sws/Unlimited.txt, available on our webpage.

1. **Constant per capita (also known as unlimited or exponential) growth**

   This question considers the unlimited growth initial value problem:

   $$R' = 0.1R; \qquad R(0) = 2{,}000$$

   (R is in rabbits; R' is in rabbits per month).

   Run the program Unlimited.sws, which will produce, on the same set of axes:

   (i) A graph (in blue circles) of an approximate solution (obtained using Euler's method) to the above initial value problem; and

   (ii) A graph (in red) of the exact, closed form solution to the above initial value problem.

   Answer these questions:

   (a) Write down the formula for the exact solution to the above IVP. (Use what you know about the unlimited growth IVP, and/or refer to the Sage code.)

   $$R = 2{,}000e^{0.1t}.$$

   (b) What *single parameter* in your program Unlimited.sws would you change, to make the numerical and closed-form solutions agree more closely? Go ahead and make that change to your program, and run it again to make sure things worked. Continue to adjust as necessary until the two solutions appear to fit each other as closely as possible.

   Decrease the stepsize. Putting stepsize=0.01 seems to do the trick.

(c) What happens to a population of 2,000 rabbits after 6 months, and after 2 years? (Use your exact solution; check your answers against what you see on the graph.)

$R(6) = 2{,}000e^{0.1 \cdot 6} = 3{,}644.24$ rabbits; $R(24) = 2{,}000e^{0.1 \cdot 24} = 22{,}046.35$ rabbits.

(d) How long does it take the rabbit population to reach 25,000? (Again, use your exact solution, and check your answer against what you see on the graph.)

$$25{,}000 = 2{,}000e^{0.1t}$$

$$\frac{25{,}000}{2{,}000} = e^{0.1t}$$

$$\ln\left(\frac{25{,}000}{2{,}000}\right) = 0.1t$$

$$t = \frac{\ln\left(\frac{25{,}000}{2{,}000}\right)}{0.1} = 25.26 \text{ months.}$$

2. Logistic growth

The following questions concern a rabbit population described by the logistic model

$$R' = 0.1R\left(1 - \frac{R}{25{,}000}\right)$$

rabbits per month. For this exercise, you will need to *modify* the program Unlimited.sws, so that it solves this logistic growth problem, instead of the unlimited growth problem on the previous page. Here are some hints for doing the modification:

- ☐ Before you do any editing, copy the code from the Unlimited.sws program, and save it under a new name – call it Logistic.sws. (Use the "Copy worksheet" and "Rename worksheet" commands under the "File..." menu at the top left of your Sage worksheet.)
- ☐ You'll need to change the quantity `tfin`, so that your graph goes out far enough in time to see what happens after 5 years.
- ☐ You'll need to add a line that specifies the carrying capacity. (That is, add a line of the form `b=...`)
- ☐ You'll need to change the formula for `Rprime` to model logistic growth instead of unlimited growth. Your new formula should involve the parameter b as well as the parameter k.
- ☐ You'll need to delete the line

  ```
  Tplot=plot(2000*e^(0.1*x),0,30,color='red',thickness=3)
  ```

  from your program.
- ☐ You'll need to delete the part that says `+Tplot` from the last line of your program.

(Your program Logistic.sws should retain the same stepsize that you ended up with in exercise 1(b) above.) Once you have made the above modifications, run the program Logistic.sws. You should get a graph that looks like the logistic growth curves we have discussed in class.

Answer these questions:

(a) Under this logistic growth model, what happens to a population of 2,000 rabbits after 6 months, after 2 years, and after 5 years? Read these values off the graph as well as you can.

From the graph we find that, very roughly, $R(6) = 3{,}500$ rabbits; $R(24) = 12{,}000$ rabbits; $R(60) = 24{,}000$ rabbits.

(b) Edit your program Logistic.sws so that your starting number of rabbits is now 40,000 instead of 2,000. Compare your new graph to the previous logistic graph. How do the graphs differ? In what ways are they similar?

The new graph is quite different from the previous one. In particular, the new R is decreasing, whereas the previous R was increasing. The two graphs are similar in that, in both cases, R levels off at the carrying capacity $b =$25,000.

(c) Fill in the blanks: In a logistic growth situation, the population will increase if the initial population is ___less___ than the carrying capacity, and will decrease if the initial population is ___larger___ than the carrying capacity. This makes sense because the ___carrying capacity___ tells us how large a population the environment can support. It also makes mathematical sense because, if the initial population is smaller than the carrying capacity, then the quantity R/b will initially be ___less___ than 1, so the derivative

$$R' = R\left(1 - \frac{R}{b}\right)$$

will initially be positive, whereas, if the initial population is ___larger___ than the carrying capacity, then the quantity R/b will initially be greater than 1, so R' will initially be ___negative___.