M408R Worksheet 1: Prediction using SIR models

Tuesday, September 2

Goal: To use the SIR model to predict (approximately) the evolution of a disease

Recall: we saw, in class this week that the spread of a disease can, under appropriate assumptions, be modeled by the SIR equations

$$\begin{aligned} S' &= -aSI, \\ I' &= aSI - bI, \\ R' &= \phantom{-aSI - {}} bI. \end{aligned}$$

Here S, I, and R denote the number of individuals susceptible, infected, and recovered, respectively, at any given time t. We agree that t is measured in days, and that S, I, R are measured in people. Then S', I', and R', which denote the rates of change of S, I, and R respectively, are measured in people per day. Finally, a and b are positive constants, called the *transmission coefficient* and the *recovery coefficient*, respectively.

Question 1. What are the *units* for a and for b? Please explain.

Now, to make things concrete, we will choose particular values for a and b, let's say $a = 0.00002$ and $b = 0.1$.

Question 2. Once you get the disease, for how long do you stay infected before you recover? Please explain.

OK, so now, and for the rest of this tutorial, the above SIR equations will look like this:

$$\begin{aligned} S' &= -0.00002SI, \\ I' &= 0.00002SI - 0.1I, \\ R' &= \phantom{-0.00002SI - {}} 0.1I. \end{aligned}$$

If we're going to predict how the disease evolves, we'd better know how it starts. So let's assume it starts with 95,000 susceptible, 5,000 infected, and none recovered.

Question 3. How large is the overall population we're working with? Please explain.

We are now going to try and construct a *table* of (approximate) values of S, I, R, S', I', and R', over the course of a few days. Here's the start of that table.

Estimates over the first four days

t	S	I	R		S'	I'	R'
0	95,000	5,000	0				
2							
4							

Question 4. In the above table, fill in the values of S', I', and R' at $t = 0$. (Use the SIR equations at the bottom of the first page of this tutorial.)

Question 5. Fill in the blanks: Suppose we want to know (or approximate) $S(2)$, meaning the value of S at $t = 2$. Since we *know* $S(0)$ (meaning the value of S at $t = 0$), all we have to do, to figure out $S(2)$, is to figure out the total *change* in S from $t =$________ to $t =$________. Let's call this change ΔS (pronounced "delta S"). Then we know (fill in the blank with a NUMBER, taken from your table above) that

$$S(2) = S(0) + \Delta S = \underline{\qquad\qquad} + \Delta S.$$

Question 6. Fill in the blanks again: Let's suppose that the *rate of change* S' of S doesn't vary too much over the first two days. Then we can assume that this rate of change over the first two days is equal to the value of S' at the *start* of these two days. That is, we can assume that, over the first two days, S' is roughly equal to (fill in the blank with a NUMBER, taken from your table above)

$$S'(0) = \underline{\qquad\qquad} \text{ people per day.}$$

NOW: note that the total change ΔS, over the first two days, can be computed by taking the *rate of change of S per day*, over those two days, and multiplying it by Δt, where Δt denotes the elapsed time (in days). In other words, we have, in the present situation,

$$\Delta S = S'(0) \times \Delta t = \underline{\qquad\qquad} \times \underline{\qquad\qquad} = \underline{\qquad\qquad}.$$

(Fill in the blanks with numbers.)

Question 7. Using your answers to the previous two questions, fill in the (approximate) value of S at $t = 2$, in the table near the top of this page.

Question 8. Explain why the value you just filled in is (probably) only an *approximate*, and not an *exact*, value.

Question 9. Fill in the blanks, again: By arguing as in questions 5–7 above, we can find that, roughly,

$$\begin{aligned} I(2) &= I(0) + \Delta I \\ &= I(0) + (\_\_\_\_\_\_\_\_ \times \Delta t) \\ &= \_\_\_\_\_\_\_\_ + (\ \_\_\_\_\_\_\_\_ \times \_\_\_\_\_\_\_\_\ ) = \_\_\_\_\_\_\_\_. \end{aligned}$$

(The last four blanks should be filled in by numbers; the blank before that, by a symbolic quantity.) Put your final answer from here into the correct place in the above table.

Question 10. Use the same kind of procedure as above to find an approximate value of R at $t = 2$:

$$\begin{aligned} R(2) &= R(0) + \Delta R \\ &= R(0) + (\_\_\_\_\_\_\_\_ \times \Delta t) \\ &= \_\_\_\_\_\_\_\_ + (\ \_\_\_\_\_\_\_\_ \times \_\_\_\_\_\_\_\_\ ) = \_\_\_\_\_\_\_\_. \end{aligned}$$

Put your final answer here into the correct place in the above table.

Question 11. Now that you have approximate values for $S(2)$, $I(2)$, and $R(2)$, you can compute approximate values of $S'(2)$, $I'(2)$, and $R'(2)$, using the SIR equations near the bottom of the first page. *Do it*, and put your numbers into the table near the top of the previous page.

Question 12. Now that you know (approximately) $S(2)$, $I(2)$, $R(2)$, *and* $S'(2)$, $I'(2)$, and $R'(2)$, you can compute (approximately) $S(4)$, $I(4)$, and $R(4)$ at $t = 4$, by repeating the above procedure. For example, you can observe that, roughly, if ΔS now denotes the change in S from $t = 2$ to $t = 4$, then

$$\begin{aligned} S(4) &= S(2) + \Delta S \\ &= S(2) + (\_\_\_\_\_\_\_\_ \times \Delta t) \\ &= \_\_\_\_\_\_\_\_ + (\ \_\_\_\_\_\_\_\_ \times \_\_\_\_\_\_\_\_\ ) = \_\_\_\_\_\_\_\_. \end{aligned}$$

(The last four blanks should be filled in by numbers; the blank before that, by a symbolic quantity.) Put your final answer here into the correct place in the above table. Repeat this procedure for I and R.

Now suppose we had taken things *one day*, instead of *two days*, at a time. That is: suppose we had taken $\Delta t = 1$. We would then compute $S(1), I(1), R(1), S'(1), I'(1)$, and $R'(1)$, based on the values of these quantities at $t = 0$, and then use the values at $t = 1$ to compute the values at $t = 2$, and so on.

Question 13. How would the values at $t = 2$, computed using $\Delta t = 1$, compare to those you computed previously, using $\Delta t = 2$? Which method is likely to give a better approximation to the *true* nature of the disease after two days? Please explain. (You don't have to actually compute anything.)