GOAL: To explore the product rule in various ways

Today's question: is the derivative of a product equal to the corresponding product of the derivatives? That is: is it true, in general, that

$$\frac{d}{dt}[f(t)g(t)] = f'(t)g'(t)? \qquad (*)$$

1. Let's investigate (∗) by means of an example. Let $f(t) = t$ and $g(t) = t^2$.

   (a) Calculate $\frac{d}{dt}[f(t)g(t)]$, i.e. first calculate the product $f(t)g(t)$, and then take the derivative.

   (b) Calculate $f'(t)g'(t)$; that is, first calculate the individual derivatives, then take the product.

   (c) For these functions $f(t)$ and $g(t)$, is equation (∗) true?

2. We are now going to think about the derivative of a product – that is, about things like $\frac{d}{dt}[f(t)g(t)]$ – intuitively, as follows.

   Imagine a rectangular oil slick (OK, not realistic, but bear with me), whose baselength and height vary with time t – say the baselength is $f(t)$, and the height is $g(t)$.

   Fill in the blanks: The *area* of the oil slick is then the product ________________. So to understand $\dfrac{d}{dt}[f(t)g(t)]$, we should try to understand the rate of ________________ of this area!

   To do so, suppose the *height* $g(t)$ of the rectangle changes by a small amount. Then the overall change in the *area* of the rectangle will be relatively LARGE if the baselength $f(t)$ of the rectangle is large, and will be relatively SMALL if the baselength $f(t)$ of the rectangle is ________________.

   Similarly, suppose the *baselength* ________________ of the rectangle changes by a small amount. Then the overall change in the *area* of the rectangle will be relatively ________________ if the height $g(t)$ of the rectangle is large, and will be relatively SMALL if the ________________ $g(t)$ of the rectangle is ________________.

   In general, then, the change in the area of the rectangle will depend not only on the amounts by which the height $g(t)$ and baselength ________________ change, but also on the *magnitudes* of $f(t)$ and ________________. Therefore, the *rate of change* $d[f(t)g(t)]/dt$ of the ________________ of the rectangle will depend not only on the rates of change $f'(t)$ and ________________, but also on the quantities ________________ and $g(t)$ themselves.

3. Spoiler alert!! It turns out that the *correct* formula for the derivative of a product is given by the following, called *the product rule* (go figure!!):

$$\frac{d}{dt}[f(t)g(t)] = f(t)g'(t) + g(t)f'(t).$$

   Explain how this formula fits in with the discussion of exercise 2 above.

4. Use the *product rule* to find $\frac{d}{dt}[f(t)g(t)]$, where $f(t)$ and $g(t)$ are again as in exercise 1 above. Does your answer agree with the result of exercise 1(a)?

5. Product rule practice: find

   (a) $\frac{d}{dx}[x\sin(x)]$.

   (b) $k'(x)$ if $k(x) = x^2 \cdot 2^x$.

   (c) $\frac{d}{dz}[\sin(z)\cos(z)]$. Use the trig identity $\cos^2(z) - \sin^2(z) = \cos(2z)$ to simplify your answer.