Second M408R in-class problem, November 19, 2014

Consider the region in the plane bounded by the curve $y = \ln(x)$, the line $x = 1$, and the line $y = 1$. Note that the corners of this region are at $(1, 0)$, $(1, 1)$ and $(e, 1)$. Find the area of this region.

Solution:

The obvious way to find areas is by slicing the region into vertical strips. Each strip has width Δx and height $1 - \ln(x)$ (since the top of the strip is at $y = 1$ and the bottom is at $y = \ln(x)$, so the strip has approximate area $(1 - \ln(x))\Delta x$. Adding up the strips gives

$$\sum_{i=1}^{n} (1 - \ln(x_i^*))\Delta x,$$

where $\Delta x = (e - 1)/N$ and x_i^* is a sample point chosen somewhere between $x_{i-1} = 1 + (i - 1)\Delta x$ and $x_i = 1 + i\Delta x$. Taking a limit as $N \to \infty$ gives the integral

$$\int_1^e (1 - \ln(x))dx.$$

The trouble with this method is that we currently don't know how to find an anti-derivative of $1 - \ln(x)$, so we can't apply the Fundamental Theorem of Calculus. After Thanksgiving we'll learn a method, called Integration by Parts, that handles expressions like this, but for now we're stuck.

So instead, we can cut the region into **horizontal** strips, each of height $\Delta y = 1/N$. The left end of each strip is at $x = 1$, and the right end is on the curve $y = \ln(x)$. But $y = \ln(x)$ means the same thing as $x = e^y$, so the right endpoint is at $x = e^y$, and the total width of the strip is $e^y - 1$. This makes the area of the strip $(e^y - 1)\Delta y$. Adding up the strips gives

$$\sum_{i=1}^{N} (e^{y_i^*} - 1)\Delta y,$$

where $\Delta y = 1/N$, $y_i = 0 + i\Delta y$ and y_i^* is a sample point chosen somewhere between y_{i-1} and y_i. This is **exactly** the same sort of setup as before, only with a different function and with the variable being y instead of x. The limit as $N \to \infty$ of this sum is the integral

$$\int_0^1 (e^y - 1)dy.$$

That's something that we **do** know how to compute. An anti-derivative of $e^y - 1$ is $e^y - y$. (Remember that y is our variable, so we want something whose derivative **with respect to** y is $e^y - 1$). Plugging in we get

$$\text{Area of } T = \int_0^1 (e^y - 1)dy = e^y - y\Big|_0^1 = (e - 1) - (1 - 0) = e - 2 \approx 0.71828.$$