Six Pillars of Calculus

by Lorenzo Sadun

Calculus is generally viewed as a difficult subject, with hundreds of formulas to memorize and many applications to the real world. However, almost all of calculus boils down to six basic ideas, together with one precalculus idea. If you really understand these ideas, the formulas will come easily. If you don't understand these ideas, the formulas won't make any sense.

We'll develop some of the ideas slowly. In M408N we'll cover ideas 1-5. Chapter 2 of the text is all about Idea 1. Chapter 3 is Idea 2. Chapter 4 is Idea 3. In Chapter 5 we get Ideas 4 and 5. M408S starts with Idea 5 and runs with it. M408L is all about Idea 6. But before we get bogged down in the details, let's see what the ideas actually are.

Precalculus idea: A picture is worth 1,000 words

When trying to understand a physical quantity, graph it! The shape of the graph will tell you many things about the underlying quantity.

Suppose we have a graph $y = f(x)$, and you're given geometric questions like "what's the slope of the tangent line to this curve?" or "what's the equation of the tangent line" or "what's the area under the curve between a and b?" Your first reaction is probably "who **cares**?" Nobody walks around town computing areas and slopes!

On the other hand, suppose that you're in business, and that the graph gives the profits of your company as a function of time. High values mean a good year; negative values mean that you lost money. The slope of the curve is the trend; positive slope means things are looking up, and negative slope means that things are getting worse. If you had to predict how things will go for the next year or two, you'd probably extend the curve in a straight line. That's the tangent line! The area under the curve, between 2003 and 2009, is the amount of money that the company made in that time. Once you have context, the answers to the mathematical questions really matter.

1 Close is good enough

In most areas of math, the goal is to get an exact answer directly. The solutions to $x^2 - 3x + 2 = 0$ are $x = 1$ and $x = 2$; you get them either from the quadratic formula or by factoring $x^2 - 3x + 2 = (x - 1)(x - 2)$. In many cases, however, it is impossible to get an exact answer directly. Instead, we get an approximate answer, then a better answer, then a better answer. The exact answer is the *limit* of these approximations.

For instance, suppose we want to sum $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ on forever. We add up the first two numbers and get $1\frac{1}{2}$. We add up the first three and get $1\frac{3}{4}$. We add up the first four and get $1\frac{7}{8}$. The more terms we add, the closer we get to two. None of these finite sums gives exactly the right answer — they all fall a little short — but taken together, they give two as the limit.

All of calculus is based on this principle!! A derivative, or instantaneous rate of change, is the limit of an average rate of change. An integral is the limit of an approximating sum. Infinite series such as $1 + \frac{1}{2} + \cdots$ are limits of finite sums.

2 Track the changes

You probably have used the "track changes" feature on Word when editing a document. If you know what has changed, you don't have to re-read the whole document. If you know who made the changes, and when, then you know what's going on. The same thing goes for functions. Instead of looking at the function itself, you can learn a lot by studying how the function is changing. Here's how we do it.

Imagine that you are driving down the road with a broken speedometer, and you want to know how fast you are going. Fortunately, you have mile markers by the side of the road, and you have a passenger with a stopwatch and a calculator.

If you take the distance traveled over the last hour and divide by one hour, you'll get your average speed for the past hour. That may be a reasonable estimate of your speed right now, but it doesn't take into account how much you've sped up or slowed down in the last hour. Taking the distance traveled for the last minute, divided by one minute, will give a better estimate, since you probably haven't changed speed much in the last minute. Averaging over the last second will do even better. Averaging over a millisecond is better still. Your *instantaneous speed* is the limit of your average speed as you average over shorter and shorter time intervals.

More generally, the rate of change of any function $f(x)$ is called the *derivative* of f, and is denoted df/dx or $f'(x)$. It is the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

of the change in $f(x)$ divided by the change in x over shorter and shorter intervals of size h.

Fortunately, we can use algebra to help us compute this limit. We can find useful formulas for the derivatives of common functions (such as $dx^n/dx = nx^{n-1}$), for the derivatives of products and ratios of functions, and for the derivatives of compound functions (such as $\sin(x^2)$). These formulas are worth knowing, and worth memorizing, but they should never take the place of understanding what a derivative *is*.

3 What goes up has to stop before coming down

If you toss a ball in the air, how fast is it rising when it reaches the top of its arc? The answer can't be positive, or else the ball would still be rising, and would be even higher an instant later. The answer can't be negative, or else the ball would already be falling, and would have been even higher an instant earlier. Since the answer isn't positive or negative, it must be zero.

In general, the maximum of a smooth function can only occur at an endpoint or where the derivative is zero. By studying derivatives of functions, and where these derivatives are zero, we can solve all sorts of real-world optimization problems.

In fact, the word *calculus* comes from the Latin title of a paper by Leibniz: "A new method for maxima and minima as well as tangents, which is neither impeded by fractional nor irrational quantities, and a remarkable type of calculation (*calculus*) for them"

4 The whole is the sum of the parts

Everybody knows that the area of a rectangle is width times height, but how do you find the area of an irregular region? For instance, how do you find the area under the curve $y = x^2$ from $x = 0$ to $x = 1$? The answer is to chop the region into thin vertical strips of width Δx, and whose height at x is x^2. Each strip is practically a rectangle of area $x^2 \Delta x$, so the total area of the region is very close to $\sum x^2 \Delta x$, where the Greek letter $\sum$ denotes "sum". The exact sum is the limit of this procedure as we slice finer and finer. It is called the *integral* of $f(x)$, and is denoted $\int_0^1 x^2 dx$.

The same idea works for finding the volume of a solid, or the moment of inertia of a rod, or the total electric charge in a region. Whenever you want to compute a bulk quantity, break it into pieces that you know how to handle, estimate the contribution of each piece, add them all up, and take a limit.

5 The whole change is the sum of the partial changes

The change in a function between now and next year is the change between today and tomorrow, plus the change between tomorrow and the day after, and so on. In other words, the total change in a quantity can be gotten by summing (or, in the limit, integrating) the tiny day-to-day changes.

This is called the *Fundamental Theorem of Calculus*, and comes in two versions. The first version says that the change in a function is the integral of its derivative. If $F'(x) = f(x)$, then $\int_a^b f(x)dx = F(b) - F(a)$. The second version says that the derivative of the integral is the original function: $\frac{d}{dx} \int_a^x f(t)dt = f(x)$.

Evaluating $\int_a^b f(x)dx$ is almost always done by finding a function $F(x)$ whose derivative is $f(x)$, and then evaluating at b and a. For instance, the derivative of $x^3/3$ is x^2, so $\int_0^1 x^2 dx = 1^3/3 - 0^3/3 = 1/3$.

There are *many* tricks for finding $F(x)$ given $f(x)$, such as integration by substitution, integration by parts, integration by partial fractions, and trig substitutions. These tricks are useful and worth knowing, but they are just a bag of tricks, mostly obtained by taking all the rules for derivatives and turning them inside out.

6 One variable at a time

When doing multi-variable calculus, almost all calculations can be done one variable at a time, treating all the other variables as constants.

For instance, suppose you have a function $f(x, y, z)$. We can hold y and z fixed, and take a derivative with respect to x. This is called a *partial derivative* and is denoted $\frac{\partial f}{\partial x}$. It measures how fast the function changes if you vary x but keep y and z fixed. Likewise, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ measure what happens when you change only y, or only z.

If you change x, y and z simultaneously, say by amounts Δx, Δy and Δz, you just have to add the contributions from changing each variable. The total change in the function is approximately $\frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial z}\Delta z$. Each term is what you would get if you only changed one variable, and the sum is what you get if you change all three.

Integrals can also be done one variable at a time. Suppose you want the volume between the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$. Chop the region into little blocks of size Δx by Δy by Δz, hence volume $\Delta x \Delta y \Delta z$. For each value of x and y, add up the volumes of the blocks over (x, y). These blocks form a tower, whose volume is a sum over z, In the limit as $\Delta z \to 0$ it becomes an integral $\Delta x \Delta y \int_0^{1-x^2-y^2} 1 \; dz$. Then, for fixed x, add up the volumes of the towers over different values of y. This yields an integral over y. Finally, add up the slices that you got for different values of x. The end result is an integral over x of an integral over y of an integral over z, namely

$$\int_{-1}^{1} \left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(\int_0^{1-x^2-y^2} 1 \; dz \right) dy \right) dx.$$