M408N First Midterm Exam Solutions, September 27, 2011

1) Compute $\sec(\sin^{-1}(3/5))$. In other words, if $\sin(\theta) = 3/5$ and $-\pi/2 < \theta < \pi/2$, what is $\sec(\theta)$?

Imagine a right triangle with base 4, height 3 and hypotenuse 5. The angle whose sine is 3/5 has cosine 4/5 and secant 5/4, so the answer is 5/4.

2) If $e^{3\ln(x)} = 8$, what is x? Simplify your answer as much as possible.

Since $e^{ln(x)} = x$, $e^{3\ln(x)} = (e^{\ln(x)})^3 = x^3$, so $x^3 = 8$, so $x = 2$.

3. Compute $\lim\limits_{x\to 3} \dfrac{x^2 - x - 6}{x - 3}$.

Since $x^2 - x - 6 = (x-3)(x+2)$, $\lim_{x\to 3} \frac{x^2-x-6}{x-3} = \lim_{x\to 3} x + 2 = 5$. You can also get this answer by realizing that the limit is (by definition!) the derivative of $x^2 - x - 6$ at $x = 3$, and using the formula for the derivative of a polynomial.

4. Let $f(x) = \sqrt{2x^2 + 1}$, with a domain of $x \geq 0$. Find the formula for the inverse function $f^{-1}(x)$.

If $y = \sqrt{2x^2 + 1}$, then $y^2 = 2x^2 + 1$, so $x = \sqrt{\frac{y^2-1}{2}} = f^{-1}(y)$. Thus $f^{-1}(x) = \sqrt{\frac{x^2-1}{2}}$.

5. Consider the function $f(x) = \frac{x^2-1}{x^2-4}$. Find the vertical and horizontal asymptotes and sketch the graph $y = f(x)$.

The function blows up where the denominator is zero (and the numerator isn't), namely at $x = \pm 2$, so we have vertical asymptotes at $x = \pm 2$. The limits as $x \to \infty$ and $x \to -\infty$ are both 1, since that's the limit of x^2/x^2, so we have a horizontal asymptote at $x = 1$. I don't know how to draw the graph on screen, but here's a description. The curve starts just above $y = 1$ on the left hand side, and rises to a vertical asymptote at $x = -2$. The limit as $x \to -2^-$ is $+\infty$, since both $x^2 - 1$ and $x^2 - 4$ are positive (and so is the limit as $x \to 2^+$.) The limit as $x \to -2^+$ is $-\infty$, as is the limit as $x \to 2^-$, since in those cases $x^2 - 1$ is positive and $x^2 - 4$ is negative. So the graph comes up from $-\infty$ near $x = -2$, rises to a maximum of 1/4 at $x = 0$, and then drops down through the floor as x approaches 2. For $x > 2$, the graph comes down through the ceiling and approaches the horizontal asymptote of $y = 1$ from above.

6. Consider the function $f(x) = \begin{cases} x^2 & x > 3 \\ 3x & x \leq 3 \end{cases}$. Is $f(x)$ continuous? Why or

why not?

The only place where it might fail to be continuous is at $x = 3$, so we compute: $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} 3x = 9$, $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} x^2 = 9$, and $f(3) = 3(3) = 9$. Since these all agree, $f(x)$ is continuous at 3, and is continuous everywhere. Note that there are THREE quantities to be compared: the limit as $x \to 3^+$, the limit as $x \to 3^-$, and the actual value at 3. Just comparing the two limits isn't enough – their being equal shows that an overall limit exists, but doesn't prove continuity, since there could (in principle) be a removable discontinuity there.

7. Suppose the position of a particle at time t is given by the function $f(t) = 2^{-t}$. (a) Graph position versus time on the first blank piece of graph paper. Be as precise as possible.

The graph of position should go through (-2,4), (-1,2), (0,1), (1,1/2), (2,1/4), (3,1/8), and approach the x axis to the right. This looks just like the graph of $y = 2^t$, only reflected across the y axis. Note that the curve is positive everywhere, and is decreasing everywhere.

(b) Sketch a graph of *velocity* versus time on the second blank piece. This graph is *not* expected to be precise, but should be qualitatively right. You do *not* need the formula for the derivative of 2^{-t} to do this! Instead, I expect you to graph the derivative of $f(t)$ based on the shape of the graph of $f(t)$.

Since the position is decreasing, the velocity is always negative. (Put another way, the tangent line is always sloping downwards, so the derivative is negative.) It is large and negative when $t < 0$, and is smaller and negative when $t > 0$. In fact, the graph of velocity looks more-or-less like the the graph of position, just turned upside-down (and shrunk vertically a little bit). We will soon learn that the actual derivative of 2^{-t} is $-\ln(2)2^{-t} \approx -0.69(2^{-t})$.

8. Consider the function $f(x) = 5^x$. Which of the following expressions are equal to $f'(2)$? Circle *all* correct expressions — there may not be any, there may be one, or there may be more than one. For this problem (and *only* for this problem), explanations are unnecessary and will not be considered in the grading.

a) $25 \lim_{h\to 0} \dfrac{5^h - 1}{h}$

This is correct, being the $h \to 0$ definition of a derivative, with the factor of 5^2 separated out. More explicitly,

$$f'(2) = \lim_{h\to 0} \frac{5^{2+h} - 5^2}{h} = 5^2 \lim_{h\to 0} \frac{5^h - 1}{h}$$

b) $2(5)^{2-1}$

This is dead wrong. The derivative of 5^x is NOT $x5^{x-1}$.

c) $\lim_{x\to 2} \frac{5^x - 25}{x-5}$.

This is wrong. The $x \to a$ definition of a derivative should have $x-2$, rather than $x-5$, in the denominator.

d) The slope of the line tangent to $y = f(x)$ at $(2, 25)$.

This is correct, being one of the standard applications of a derivative.

9. Let $f(x) = 1/x$. Compute $f'(-4)$ **FROM THE DEFINITION OF THE DERIVATIVE AS A LIMIT**, making clear what you are doing at every step. (If you just plug into the formula for the derivative of x^n you will not get any credit.)

$f'(-4) = \lim_{x\to -4} \frac{f(x)-f(-4)}{x-(-4)} = \lim_{x\to -4} \frac{\frac{1}{x}+\frac{1}{4}}{x+4}$. You already did this one in homework! The numerator is $(x+4)/4x$, so the ratio is $1/4x$, which approaches -1/16 as $x \to -4$.

You could also do this as an $h \to 0$ limit, getting $f'(x)$ as:

$$\lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h} == \lim_{h\to 0} \frac{-h/(x(x+h))}{h} = \lim_{h\to 0} \frac{-1}{x(x+h)} = \frac{-1}{x^2},$$

and then plugging in $x = -4$. Note that $x - (x+h) = -h$ is NOT the same as $(x-x)+h = h$. A lot of people wrote "$x - x + h$" without parentheses and made that mistake!

10. Suppose that $f(3) = 4$ and $f'(3) = -1$. Find the equation of the tangent line to $y = f(x)$ at $(3, 4)$.

The tangent line has slope -1 and goes through (3,4), and hence is $y - 4 = -(x-3)$, or $y = 7 - x$.

For extra credit, use this tangent line to approximate $f(3.05)$.

Since the actual graph hugs the tangent line (and vice-versa), $f(3.05) \approx 7 - 3.05 = 3.95$.