M408N Second Midterm Exam solutions, October 25, 2012

1. (42 points) Compute dy/dx in each of these situations. You do not need to simplify:

a) $y = \tan^{-1}(2x)$

By the chain rule, $\frac{dy}{dx} = \frac{2}{1+(2x)^2} = \frac{2}{1+4x^2}$.

b) $y = \frac{\ln(x)+5}{\sin(x)+2}$

By the quotient rule, $\frac{dy}{dx} = \frac{(\sin(x)+2)(\frac{1}{x})-(\ln(x)+5)\cos(x)}{(\sin(x)+2)^2}$

c) $y = \ln(xe^x + 1)$

This uses both the chain rule and the product rule: $\frac{dy}{dx} = \frac{\frac{d}{dx}(xe^x+1)}{xe^x+1} = \frac{xe^x+e^x}{xe^x+1}$

d) $y = \sin(x)\sin^{-1}(x)$

This is the product rule: $\frac{dy}{dx} = \cos(x)\sin^{-1}(x) + \frac{\sin(x)}{\sqrt{1-x^2}}$.

e) $y = x^{3x}$

Use logarithmic differentiation. $\ln(y) = 3x\ln(x)$, so $(\ln(y))' = 3\ln(x)+3$, so $\frac{dy}{dx} = (3\ln(x) + 3)x^{3x}$.

f) $y = e^{\sin(\cos(\ln(x)))}$

Apply the chain rule 3 times: $\frac{dy}{dx} = e^{\sin(\cos(\ln(x)))}\cos(\cos(\ln(x)))(-\sin(\ln(x)))(1/x)$

g) $xe^y = \ln(xy + 1) + 8$

$e^y + xe^y y' = \frac{y+xy'}{xy+1}$, so $e^y(xy + 1) + xe^y(xy + 1)y' = y + xy'$, so $[xe^y(xy + 1) - x]y' = y - e^y(xy + 1)$, so $\frac{dy}{dx} = y' = \frac{y-e^y(xy+1)}{xe^y(xy+1)-x}$.

2) (12 pts) Derivatives and limits

a) Let $f(x)$ be a function that is differentiable at $x = a$. State the definition of $f'(a)$ as a limit. [There are two standard forms – either one will do.]

$$f'(a) = \lim_{h\to 0} \frac{f(a + h) - f(a)}{h} = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}.$$

b) Compute $\lim_{h\to 0} \frac{\sin(\pi(1 + h)^2) - \sin(\pi)}{h}$

This is the derivative of $f(x) = \sin(\pi x^2)$ at $x = 1$. Since $f'(x) = 2x\pi\cos(\pi x^2)$, $f'(1) = 2\pi\cos(\pi) = -2\pi$.

c) Compute $\lim_{x\to 5} \frac{x^2 e^x - 25e^5}{x - 5}$

This is the derivative of $f(x) = x^2 e^x$ at $x = 5$. Since $f'(x) = (x^2 + 2x)e^x$,

then answer is $35e^5$.

3. (12 points) Consider the function $f(x) = e^{\sin(3x)}$.

a) Find the equation of the line tangent to the curve $y = f(x)$ at $x = 0$.

$f'(x) = 3\cos(x)e^{\sin(3x)}$, so $f'(0) = 3$. Since $f(0) = 1$, we have $y - 1 = 3(x - 0)$, or $y - 1 = 3x$, or $y = 3x + 1$.

b) Find the linearization of the function $f(x) = e^{\sin(3x)}$ around $a = 0$.

This is the same question! $L(x) = 3x + 1$.

c) Using either differentials of linear approximations, approximate the value of $e^{\sin(0.3)}$.

We are computing $f(0.1)$, which is approximately $L(0.1) = 1.3$. (The actual answer is around 1.35).

4. (12 pts) Evaluate the following trig expressions:

a) $\cos(4\pi/3)$ $4\pi/3$ radians is 240 degrees. The answer is $-1/2$.

b) $\tan(3\pi/4)$

$3\pi/4$ radians is 135 degrees, or 45 degrees past vertical. The tangent is -1.

c) $\sin^{-1}(\sin(5\pi/6))$

We are looking for an angle **between** $-\pi/2$ **and** $\pi/2$ whose sine is $\sin(5\pi/6)$. Since $\sin(\pi - x) = \sin(x)$, the answer is $\pi/6$. (Aka 30 degrees)

5. (12 pts) Related rates. A team is digging a hemispherical hole. They start by making a small hole, and enlarge it in all directions so that it always has the shape of a hemisphere. [Note: the volume of a sphere is $\frac{4}{3}\pi r^3$, where r is the radius.] They are excavating dirt at a rate of 40 cubic feet per minute. We want to understand how fast the radius of the hole is increasing.

a) Draw a picture of the situation. Label your variables. Write down an equation relating the variables.

Your picture should show a hole of radius r and volume V, where $V = \frac{2}{3}\pi r^3$ – that's half the volume of a sphere of radius r.

b) At the instant that the radius reaches 10 feet, how fast is the radius increasing?

The derivative of our relation is $\frac{dV}{dt} = 2\pi r^2 \frac{dr}{dt}$. Plugging in $r = 10$ and $dV/dt = 40$ gives $40 = 200\pi(dr/dt)$, so $dr/dr = 1/(5\pi)$ (feet per minute, since r is measured in feet and dV/dt is measured in cubic feet per minute.)

6. (10 points) Sketch the graph of the derivative of the function whose graph is shown below. I don't expect you to get the details of the derivative right, but you should clearly show where the derivative is positive, negative and zero, where it is big and where it is small. Draw your graph directly under the one shown, so that the x-values can be compared.
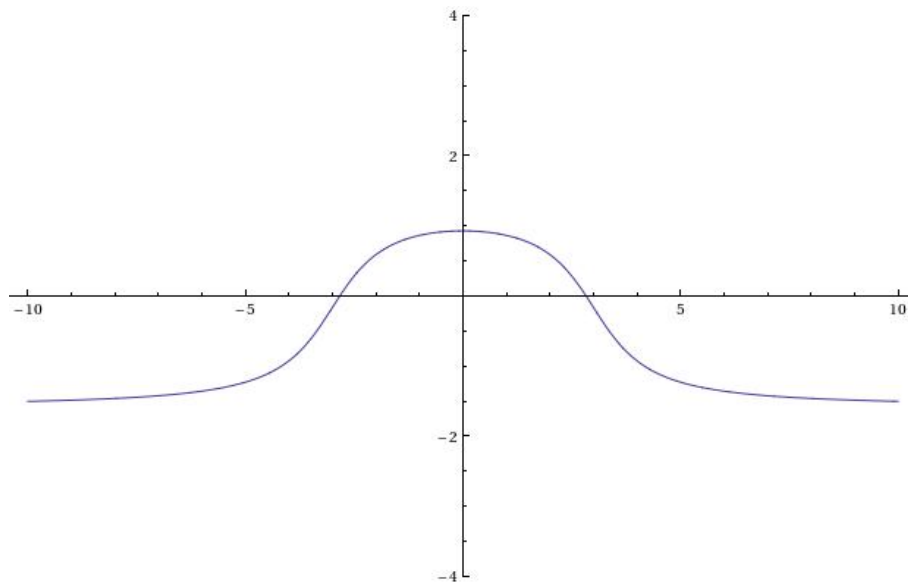
Here is the derivative:
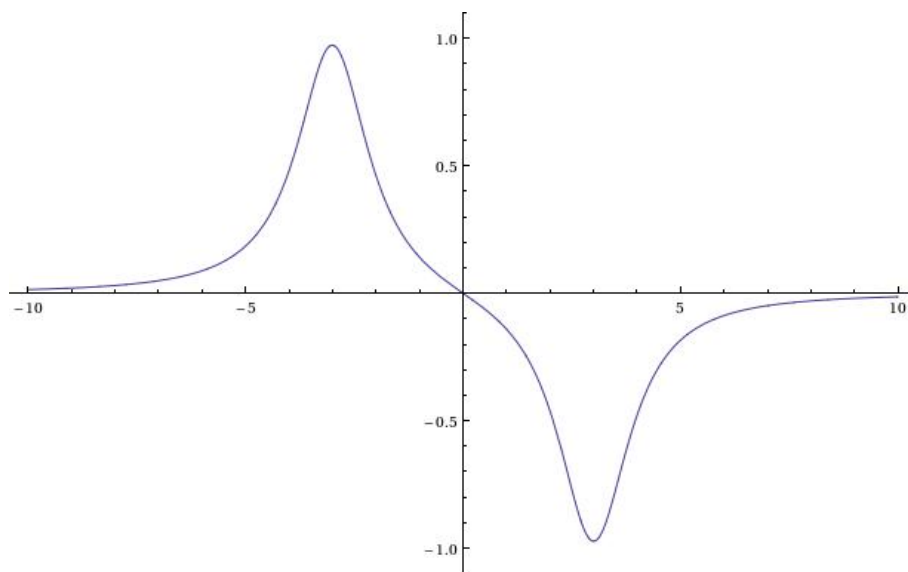
The graph of $f(x)$ starts out fairly flat, with a derivative slightly greater than zero. The slope increases until $x = -3$, and then settles back down, hitting zero when $x = 0$. When $x > 0$ the function is decreasing, meaning the derivative is negative. The curve is flat at first, then steep around $x = 3$, and then flattens out again.

By the way, the formula for the original function was $f(x) = \tan^{-1}(x+3) - \tan^{-1}(x-3) - \pi/2$, whose derivative is $f'(x) = \frac{1}{1+(x+3)^2} - \frac{1}{1+(x-3)^2}$.