M408N First Midterm Exam (with solutions), October 8, 2015

1) (15 pts) (Inverse) trig functions

a) Draw a right triangle where one of the angles is $\tan^{-1}(2)$. (There are many possible answers, all with the same shape but different overall size. Pick your favorite.) Label the lengths of all three sides and then compute $\sin(\tan^{-1}(2))$.

*One choice for this triangle would have opposite side 2, adjacent side 1, and hypotenuse* $\sqrt{2^2+1^2} = \sqrt{5}$*. By soh-cah-toa, the sine of* $\tan^{-1}(2)$ *is* $2/\sqrt{5}$*.*

b) Compute $\cos(5\pi/6)$.

*This is one of the angles that you should have memorized.* $5\pi/6$ *radians is 150 degrees, the sine is 1/2 and the cosine is* $-\sqrt{3}/2$*.*

c) Draw a right triangle where one of the angles is $\sec^{-1}(2)$. Label the lengths of all three sides and then compute $\tan(\sec^{-1}(2))$.

*This triangle has hypotenuse 2, adjacent side 1, and so opposite side* $\sqrt{2^2-1^2} = \sqrt{3}$*. The tangent of* $\sec^{-1}(2)$ *is then* $\sqrt{3}/1 = \sqrt{3}$*. You could also get this from the formula* $\tan^2(\theta) = \sec^2(\theta) - 1 = 3$*. By the way, the angle* $\sec^{-1}(2)$ *is* $\pi/6$*, or 30 degrees.*

2. (20 points) Compute the following limits:

a) $\displaystyle\lim_{x\to 4} \frac{x^2-5x+4}{x^2-16} = \lim_{x\to 4} \frac{(x-1)(x-4)}{(x+4)(x-4)} = \lim_{x\to 4} \frac{(x-1)}{(x+4)} = \frac{3}{8}$.

b) $\displaystyle\lim_{x\to\infty} \frac{x^2-5x+e^{-x}}{x^2-16+3e^{-x}} = \lim_{x\to\infty} \frac{1-5/x+e^{-x}/x^2}{1-16/x^2+3e^{-x}/x^2} = \frac{1}{1} = 1$. *This is because, as* $x\to\infty$*,* $1/x$*,* $1/x^2$ *and* $e^{-x}/x^2$ *all go to zero.*

c) $\displaystyle\lim_{x\to-\infty} \frac{x^2-5x+e^{-x}}{x^2-16+3e^{-x}}$.

*In this direction,* $e^{-x}$ *does NOT go to zero. Rather, it is the dominant term, growing (much) faster than all the other terms, so the limiting ratio is* 1/3.

d) $\displaystyle\lim_{x\to 0+} \frac{\tan(3x)}{4x}$.

*Rewrite this as* $\frac{\sin(3x)}{4x\cos(4x)}$*. As* $x\to 0$*,* $\sin(3x)/3x \to 1$*, so* $\sin(3x)/4x \to 3/4$*, while* $\cos(4x)\to\cos(0) = 1$*, so the limit is* $3/4$*.*

3. (15 pts) Continuity and discontinuities.

a) Where does the function $f(x) = \dfrac{x^2 - 9}{x^2 - 4x + 3}$ fail to be continuous?

*The ratio of polynomials is continuous everywhere except where the denominator is zero. Since the denominator factors as* $(x-3)(x-1)$*, the discontinuities are at* $x = 1$ *and* $x = 3$.

b) For each point where $f(x)$ isn't continuous, identify the kind of discontinuity.

*The numerator factors as* $(x-3)(x+3)$*. The discontinuity at* $x = 3$ *is* **removable***, since the limit as* $x \to 3$ *exist (and equals 3), insofar as the* $(x-3)$*'s in the numerator and denominator cancel. However, the function has an* **infinite discontinuity** *at* $x = 1$*, since the denominator goes to zero while the numerator does not.*

4. (15 pts) Definition of derivative.

Consider the limit

$$\lim_{h \to 0} \frac{(5+h)e^{5+h} - 5e^5}{h}.$$

a) Find a function $f(x)$ and a point $a$ such that this limit equals $f'(a)$.

*There's actually more than one answer, but the simplest one is* $f(x) = xe^x$ *and* $a = 5$*. Then we're taking the limit of* $(f(a+h) - f(a))/h$*, which is* $f'(a)$.

b) Using what you know about taking derivatives, evaluate the limit.

*By the product rule,* $f'(x) = e^x + xe^x = (x+1)e^x$*, so* $f'(5) = 6e^5$.

5. (20 pts) Compute the derivatives of the following functions with respect to $x$.

a) $(\sin(x) + 3)(e^x + x^2)$.

*By the product rule, this is* $(\cos(x))(e^x + x^2) + (\sin(x) + 3)(e^x + 2x)$.

b) $\dfrac{\sin(x) + 3}{e^x + x^2}$.

*By the quotient rule, this is* $\frac{(\sin(x)+3)(e^x+2x) - (\cos(x))(e^x+x^2)}{(e^x+x^2)^2}$.

c) $\sin(e^{5x} + x^2)$.

*By the chain rule (applied twice), this is* $\cos(e^{5x} + x^2)(5e^{5x} + 2x)$.

d) $\sin^2(x) + \cos^2(x)$.

*There are two ways to see that this is zero. One is to recognize that* $\sin^2(x) + \cos^2(x) = 1$. *The other is to just use the chain rule or product rule on each term. The derivative is* $2\sin(x)\cos(x) + 2\cos(x)(-\sin(x)) = 0$.

6. (15 pts) Implicit differentiation.

Find the equation of the line tangent to the curve $x^2 + y^3 = 9$ at the point $(1, 2)$.

*We have* $2x + 3y^2y' = 0$, *so* $y' = \frac{-2x}{3y^2}$. *At (1,2) this gives* $y' = -2/12 = -1/6$. *By the point-slope formula, the equation of our line is then* $(y-2) = -(x-1)/6$, *or equivalently* $y = \frac{13-x}{6}$, *or equivalently* $6y + x = 13$.