M408N Second Midterm Exam Solutions, November 12, 2015

1) Tangents and differentials

a) Find the equation of the line tangent to $y = x^{1/3}$ at (1000,10).

$f(x) = x^{1/3}$, so $f'(x) = (1/3)x^{-2/3}$, so the slope of the tangent line is $f'(1000) = 1/300$. By point-slope, this makes the equation of the line $y - 10 = (x - 1000)/300$.

b) Estimate $\sqrt[3]{994}$.

Plugging $x = 994$ into the equation of the line gives $f(994) - 10 \approx (-6)/300 = -0.02$, so $f(994) \approx 9.98$.

2. Critical points.

a) Find all the critical points of the function $f(x) = x^2 e^{-x^2}$.

We compute $f'(x) = (2x - 2x^3)e^{-x^2}$. Since e^{-x^2} is never zero, the critical points are where $2x - x^3 = 0$, namely $x = -1$, $x = 0$ and $x = 1$.

b) Classify which are local maxima, which are local minima, and which are neither.

You can either use the first derivative test or the second. Using the first derivative, notice that $f'(x) > 0$ when $x < -1$, $f'(x) < 0$ when $-1 < x < 0$, $f'(x) > 0$ when $0 < x < 1$ and $f'(x) < 0$ when $x > 1$, so $x = \pm 1$ must be local maxima while $x = 0$ is a local minimum.

Or you can use the second derivative test. $f''(x) = (1 - 10x^2 + x^4)$ by the product rule. Then $f''(0) = 1 > 0$ and $f''(1) = f''(-1) = -5 < 0$, so $x = \pm 1$ must be local maxima while $x = 0$ is a local minimum.

One thing you *cannot* do is simply compare the values of $f(x)$ at the different critical points. That's how you find *global* maxima and minima on an interval. On the whole line there's no guarantee that there are any maxima or minima.

3. Concavity. On what interval(s) is the function $f(x) = x^2 e^{-3x}$ concave up? On what interval(s) is it concave down? Identify the points of inflection.

Here we need to look at $f''(x)$, which works out to $(2 - 12x + 9x^2)e^{-3x}$. By the quadratic formula, this is zero at the sub-critical points $x = (2 \pm \sqrt{2})/3$. $f'' > 0$ when $x > (2 + \sqrt{2})/3$ and when $x < (2 - \sqrt{2})/3$ and $f'' < 0$ when $(2 - \sqrt{2})/3 < x < (2 + \sqrt{2})/3$. One way to see this is by using the test points $x = 0$, $x = 1$ and $x = 2$ for the three intervals, since $f''(0) = 2 > 0$, $f''(1) = -e^{-3} < 0$ and $f''(2) = 14e^{-6} > 0$. As a result, the function is concave up on $(-\infty, (2 - \sqrt{2})/3)$ and on $((2 + \sqrt{2})/3, \infty)$ and is concave down on $((2-\sqrt{2})/3, (2+\sqrt{2})/3)$.

4. Computational grab bag. Compute the following:

a) $\frac{d}{dx} x^{\sin(x)}$.

If $f(x) = x^{\sin(x)}$, then $\ln(f) = \sin(x)\ln(x)$. Then $f'/f = d(\ln(f))/dx = \cos(x)\ln(x) + \sin(x)/x$. So $f'(x) = x^{\sin(x)}(\cos(x)\ln(x) + \frac{\sin(x)}{x})$.

b) $\lim_{x\to 3} \frac{\sin(\pi x)}{x^2-9}$.

Use L'Hospital's rule, since this is a "0/0" limit.

$$\lim_{x\to 3} \frac{\sin(\pi x)}{x^2-9} = \lim_{x\to 3} \frac{\pi\cos(\pi x)}{2x} = \frac{-\pi}{6}.$$

c) $\lim_{x\to 0^+} 3x\ln(x)$

This is a "$0\cdot\infty$" indeterminate form, which we turn into an "∞/∞" form by writing it as $3\ln(x)/(1/x)$. Applying L'Hospital's rule gives the limit of $(3/x)/(-1/x^2) = -3x$, which is 0.

d) $\lim_{x\to 0^+} x^{3x}$

The log of this expression is the limit you just did in part (c). Since $\ln(L) = 0$, $L = 1$, (where L denotes the answer to this question).

5. Related rates. The radar gun that a cop uses does **not** actually measure the speed of a car. Rather it measures the rate at which the **distance from the cop to a car** is changing. This problem compares the two.

Suppose that a car is moving due north along a north-south road. There is a historical marker on the road. A cop is standing 300 feet east of the marker and points a radar gun at the car. When the car is 400 feet north of the marker, the gun reads "60 MPH". How fast is the car actually going?

Be sure to explain each step of your work clearly. Draw a picture. Define your variables. Write down an equation relating your variables. Use derivatives to relate the different rates of change. Solve the problem.

There is a right triangle between the cop, the marker, and the car. Let r be the distance from the cop to the car, and y the distance from the car to the marker. Then $r^2 = y^2 + (300 \text{ feet })^2$, so $2r(dr/dt) = 2y(dy/dt)$, or $(dy/dt) = (r/y)dr/dt$. In this case dr/dt is 60 MPH, y is 400 feet, and $r = \sqrt{400^2+300^2} = 500$ feet. Since $r/y = 5/4$, the actual speed of the car is $(5/4)60$ MPH $= 75$ MPH.

6. Optimization. New Balance has discovered that the number of MX623 shoes (my favorite!) that it sells depends on the price as follows: If they set the price at $\$x$/pair, then they can expect to sell $1,000,000 - 10,000x$ pairs of shoes per month. [In the real world, manufacturers like New Balance sell mostly to retail stores, which then sell to customers. For this problem, let's pretend that New Balance is selling directly to customers.]

a) At what price will their monthly **revenue** from this line of shoes be maximized? How high can that revenue get?

The revenue is the price times the number of shoes sold, so $R(x) = x(1,000,000-10,000x) = 1,000,000x-10,000x^2$. Since $R'(x) = 1,000,000-20,000x$, the only critical point is at $x = 50$, that is at a price of \$ 50/pair. (You can easily check that this is a maximum, not a minimum, by the second derivative test.) At that price, the monthly revenue is $R(50) = 50(500,000) = 25,000,000$ dollars.

b) Suppose that it costs \$40/pair to make and distribute these shoes. At what price will their monthly **profit** be maximized? What is that maximum profit?

If the cost is \$40/pair, and the price is $\$x$/pair, then the profit is $\$(x-40)$ per pair. Since we sell $(1,000,000 - 10,000x)$ pairs, our profit is

$$P(x) = (x-40)(1,000,000-10,000x) = -10,000x^2+1,400,000x-40,000,000.$$

Our critical point is where $0 = P'(x) = 1,400,000 - 20,000x$, so $x = 70$. At this price, the profit is $(70 - 40)(300,000) = 9,000,000$ dollars. (For the record, this make of shoes has an actual list price of \$69.95.)