M408N Final Exam, December 12, 2016

1) Grab bag (40 points). Show your work to justify your answers!

a) Compute $\lim_{x\to 2} \dfrac{x^{-2} - 2^{-2}}{x-2}$.

b) Compute $\lim_{x\to 2} \sin\left(\dfrac{2\pi(x^{-2} - 2^{-2})}{x-2}\right)$

c) Compute $\dfrac{d}{dx} \sin(\ln(e^{5x} + \tan(x)))$

d) Compute $\dfrac{d}{dx}\left(\dfrac{\ln(x)}{x^2+1}\right)$

e) Find the slope of the line that is tangent to the curve $e^x + y^2 + y = 3$ at the point $(0, 1)$.

f) Find all the vertical and horizontal asymptotes of the curve $y = \dfrac{(e^x+1)(x^2+3x)}{(e^x+2)(x^2+4x)}$.

g) Compute $f(t)$, if $f'(t) = 6t^2 - 4t + 7$ and $f(1) = 13$.

h) Compute $\dfrac{d}{dx}\displaystyle\int_{2x}^{e^x} 2\cos(t^2+5)dt$.

i) Let $f(x) = \cos(\pi \ln(x))$. Compute $\displaystyle\int_1^e f'(x)dx$.

j) Which is bigger, $\displaystyle\int_2^8 \frac{dx}{x}$ or $\displaystyle\int_5^{25} \frac{dx}{x}$, and why?

2. (12 points) Tangent lines and linear approximations. Let $f(x) = e^{(x^2-1)}(1 + 3\ln(x))$.

a) Compute $f'(1)$.

b) Find the equation of the line tangent to the curve $y = f(x)$ at $(1, f(1))$.

c) Find the approximate value of $f(1.002)$.

3. Related rates. (9 points) Annuities are contracts that are guaranteed to pay a certain amount of money per year for a certain length of time. Economists use the term "present value" to mean what you would have to pay today to buy such an annuity, and this depends on the prevailing interest rate $r$. Suppose that the present value $V$ of an annuity depends is given by the formula

$$V = 1000\frac{1 - e^{-10r}}{r}.$$

At the moment, the interest rate is $r = 0.10$, but is increasing at a rate $dr/dt = 0.02$. At what rate is $V$ changing?

4. Derivatives and accumulations and graphs. (12 points, 2 pages)

The following is the graph of a function $f(x)$.

(`The hand-drawn graph is roughly U-shaped, with zeroes at` $x = \pm 1$ `and a minimum at` $x = 0$)

a) Sketch the graph of $f'(x)$. Your sketch doesn't have to be precise, but it should be positive/negative/zero in all the right places, and increasing/decreasing/flat in all the right places.

b) Sketch the graph of the accumulation function $A(x) = \int_0^x f(t)dt$. Your sketch doesn't have to be precise, but it should be increasing/decreasing/flat in all the right places, should be concave up/concave down in the right places, and be positive/negative in more-or-less the right places.

5. Definite integrals and Riemann sums. (15 points, 2 pages) An *annulus* (Latin for "ring") is the region between two concentric circles, as with a washer. We are going to use calculus to compute the area of an annulus with inner radius 1 and outer radius 2, as shown in the figure.

`(The figure show an annulus of inner radius 1 and outer radius 2 divided into 4 concentric sub-annuli, each of thickness 1/4. It looks a lot like the logo for Target, only with the center bullseye missing.)`

We are going to break the annulus into $N$ smaller pieces, all of the same thickness, as shown in the figure with $N = 4$. (I will call the smaller pieces "rings", and save the word "annulus" to mean the whole region.) We will estimate the area of each ring, add them up, take a limit as $N \to \infty$, and get an integral that we can then evaluate.

a) In terms of $N$, how thick is each ring?

b) Approximate the area of the $k$-th ring as its circumference times its thickness. **Indicate whether you are using the circumference of the inner edge of the ring, the outer edge, or something else.** Leave your answer as an explicit function of $k$ and $N$. (E.g. something like $5\pi^2 \sin(3 + \frac{k}{N})$, although that is NOT the correct answer!)

c) Now express the total area of the annulus as a Riemann sum.

d) Write down an integral that gives the limit of your Riemann sum as $N \to \infty$.

e) Evaluate this integral. [Hint: You can check your answer by comparing it to the difference between the area of a circle of radius 2 and the area of a circle of radius 1.]

6. Maxima and minima. (12 points) Consider the function

$$f(x) = x^2 e^{-x}$$

on the interval $[-1, 4]$.

a) Find all the critical points of $f(x)$ on this interval.

b) Determine which critical points are local maxima, which are local minima, and which are neither.

c) Now find the absolute maximum and minimum values of $f(x)$ on the interval $[-1, 4]$ (and where these values occur).