M408N First Midterm Exam, September 21, 2016

1. Precalculus

a) Draw a right triangle where one of the angles is $\sin^{-1}(3/5)$. Call this angle θ. Label the lengths of all three sides.

In a 3-4-5 right triangle, the angle opposite the smallest side is $\sin^{-1}(3/5)$. (Why 4? Because $4^2 = 5^2 - 3^2$.) You could just as well use a 6-8-10 triangle, or a 30-40-50 triangle, but 3-4-5 is simplest.

b) Compute $\cos(\theta)$ and $\tan(\theta)$.

By soh-cah-toa, the cosine is 4/5 and the tangent is 3/4. You could also get these from $\cos(\theta) = \sqrt{1 - \sin^2(\theta)}$ and $\tan(\theta) = \sin(\theta)/\cos(\theta)$, but soh-cah-toa is easier.

c) If $8^x = 4(2^{(x^2)})$, what are the possible values of x?

Since $2^{3x} = 2^2 2^{(x^2)} = 2^{x^2+2}$, we must have $3x = x^2 + 2$, so $x^2 - 3x + 2 = 0$, so $x = 1$ or $x = 2$. You can check that both values do work: $2^3 = 8 = 4(2^1)$ and $2^6 = 64 = 4(2^4)$.

2. Limits. Compute the following limits:

a) $\displaystyle\lim_{x\to 5^+} \frac{x^2 - 25}{\sqrt{x^2 - 25}}$.

The answer is 0. When x is slightly greater than 5, x^2 is slightly greater than 25, so $x^2 - 25$ is positive, so $\sqrt{x^2 - 25}$ makes sense, and $(x^2 - 25)/\sqrt{x^2 - 25} = \sqrt{x^2 - 25}$, which is close to zero. Since $(x^2 - 25)/\sqrt{x^2 - 25}$ is close to zero whenever x is slightly greater than 5, $\lim_{x\to 5^+}(x^2 - 25)/\sqrt{x^2 - 25} = 0$.

Note that $\displaystyle\lim_{x\to 5^-} \frac{x^2 - 25}{\sqrt{x^2 - 25}}$, by contrast, does not exist, since $\sqrt{x^2 - 25}$ is not defined for x slightly less than 5.

b) $\displaystyle\lim_{t\to -2} \frac{t^2 + t - 2}{t^2 - 4}$.

After factoring, we have

$$\lim_{t\to -2} \frac{(t+2)(t-1)}{(t+2)(t-2)} = \lim_{t\to -2} \frac{t-1}{t-2} = \frac{-3}{-4} = \frac{3}{4}.$$

c) $\displaystyle\lim_{w\to 0^+} e^w + \ln(w)$.

When w is slightly greater than 0, e^w is close to 1 and $\ln(w)$ is a large negative number, so the sum is a large negative number. This makes the limit $-\infty$.

3. Asymptotes and continuity. Consider the function $f(x) = \dfrac{|x^3 + x|}{x^3 - x}$.

a) Find all the points where $f(x)$ is discontinuous.

The numerator is continuous, as is the denominator, so the only discontinuities are where the denominator is zero, namely at $x = -1$, $x = 0$ and $x = +1$.

b) Find all the horizontal and vertical asymptotes of the graph of $f(x)$.

There are vertical asymptotes at $x = -1$ and $x = 1$, since the denominator goes to zero there but the numerator doesn't. There is NOT a vertical asymptote at $x = 0$, since there is a power of x in both the numerator and denominator. (FWIW, there is a jump discontinuity at $x = 0$, thanks to the absolute value. The limit from the left is +1 and the limit from the right is -1.)

The limit as $x \to \infty$ is 1, and the limit as $x \to -\infty$ is -1 (since the numerator is positive and the denominator is negative of essentially the same size), so there are horizontal asymptotes at $y = -1$ and $y = 1$.

4. Suppose that a function $f(x)$ is defined and differentiable for all x, that $f(3) = 5$ and that $f'(3) = -2$. Find the equation of the line tangent to the graph of $f(x)$ at $x = 3$.

This is really a precalc problem dressed up in calculus language. Since the tangent line goes through $(3, 5)$ and has slope -2, its equation must be

$$\begin{aligned} y - 5 &= -2(x - 3), && \text{(point-slope form)} && \text{OR} \\ y &= 5 - 2(x - 3), && \text{(`microscope equation')} && \text{OR} \\ y &= -2x + 11, && \text{(slope-intercept form)} && \text{OR} \\ 2x + y - 11 &= 0. \end{aligned}$$

Any one of these is considered a correct answer. (A few people wrote the equation in point-slope form correctly but then make an arithmetic mistake in converting to slope-intercept form, getting $y = -2x + 1$ or $y = -2x - 11$, for which I took off a few points.)

5. Suppose that $f(x) = \ln(x^2)$. Which of the following expressions are equal to $f'(2)$? Circle ALL that apply, and explain WHY each expression is, or isn't, $f'(2)$. Don't forget the laws of logarithms!

a) $\lim_{h \to 0} \dfrac{\ln(4 + 4h + h^2) - \ln(4)}{h}$

YES. $\ln(4 + 4h + h^2) = \ln((2 + h)^2) = f(2 + h)$, so we are looking at $\lim_{h \to 0} \dfrac{f(2 + h) - f(2)}{h}$, which is the definition of $f'(2)$.

b) $\lim_{x \to 2} \dfrac{\ln(x)^2 - \ln(2)^2}{x - 2}$

NO. $\ln(x)^2$ is not the same thing as $\ln(x^2)$! (A majority of the class got this one wrong.)

c) $\lim_{x \to 2} \dfrac{\ln((x - 2)^2)}{x - 2}$

NO. $\ln((x - 2)^2)$ is not the same as $\ln(x^2) - \ln(2^2)$.

d) $\lim_{x \to 2} \dfrac{\ln(x^2/4)}{x - 2}$

YES. By the laws of logarithms, $\ln(x^2/4) = \ln(x^2) - \ln(4) = f(x) - f(2)$, so we have $\lim_{x \to 2} \dfrac{f(x) - f(2)}{x - 2}$, which is the other standard form of the definition of $f'(2)$.