M408N Second Midterm Exam Solutions, October 17, 2016

1. (2 pages, 40 points) Compute the following derivatives.

a) $f'(x)$, where $f(x) = e^{5x}\cos(x)$.

By the product rule, $f'(x) = 5e^{5x}\cos(x) - e^{5x}\sin(x)$.

b) The derivative of $(e^x + e^{-x})^{15}$ with respect to x.

By the chain rule, this is $15(e^x + e^{-x})^{14}(e^x - e^{-x})$.

c) $\dfrac{dg}{dt}$, where $g(t) = \dfrac{3t+5}{2t^2+7}$.

By the quotient rule, this is

$$\frac{dg}{dt} = \frac{(2t^2+7)3 - (3t+5)(4t)}{(2t^2+7)^2} = \frac{-6t^2-20t+21}{(2t^2+7)^2}.$$

d) $\dfrac{d}{dt}\ln\left(\dfrac{3t+5}{2t^2+7}\right)$.

There are two ways to do this. The easy way is to recognize that $\ln\left(\dfrac{3t+5}{2t^2+7}\right) =$ $\ln(3t+5) - \ln(2t^2+7)$, so the derivative is $\frac{3}{3t+5} - \frac{4t}{2t^2+7}$.

The hard way is to use the previous result and the chain rule, getting

$$\left(\frac{3t+1}{2t^2+7}\right)^{-1}\frac{d}{dt}\ln\left(\frac{3t+5}{2t^2+7}\right) = \frac{-6t^2-20t+21}{(3t+1)(2t^2+7)},$$

which works out to the same thing.

e) $h'(s)$, where $h(s) = s\ln(s) - s$.

By the product rule, this is $s\frac{1}{s} + 1\cdot\ln(s) - 1 = \ln(s)$.

f) $\dfrac{dy}{dx}$, where $y = u\ln(u) - u$ and $u = e^x + \tan^{-1}(x)$. Express your answer as a function of x only.

By the chain rule and the previous problem,

$$\frac{dy}{dx} = \ln(u)\frac{du}{dx} = \ln\left(e^x + \tan^{-1}(x)\right)\left(e^x + \frac{1}{1+x^2}\right).$$

g) $\dfrac{dy}{dx}$, where $y = (1+e^x)^{5x}$.

If $y = (1+e^x)^{5x}$, then $\ln(y) = 5x\ln(1+e^x)$, so $\frac{y'}{y} = 5\ln(1+e^x) + \frac{5xe^x}{1+e^x}$, so

$$y' = (1+e^x)^{5x}\left(5\ln(1+e^x) + \frac{5xe^x}{1+e^x}\right).$$

2. (20 points) Derivative fundamentals. In this problem we will compute a derivative of the function $f(x) = \sqrt{x}$ **from the definition**. Using a convenient formula for derivatives of powers will not get you **any points whatsoever**.

a) What is the definition of $f'(9)$ as a limit? (There are two correct answers. I'll accept either one.)

Either $f'(9) = \lim_{x \to 9} \frac{\sqrt{x}-3}{x-9}$ or $f'(9) = \lim_{h \to 0} \frac{\sqrt{9+h}-3}{h}$.

b) Evaluate this limit to compute $f'(9)$.

We have

$$\begin{aligned} \lim_{x \to 9} \frac{\sqrt{x}-3}{x-9} &= \lim_{x \to 9} \frac{\sqrt{x}-3}{x-9} \left(\frac{\sqrt{x}+3}{\sqrt{x}+3} \right) \\ &= \lim_{x \to 9} \frac{x-9}{(x-9)(\sqrt{x}+3)} \\ &= \lim_{x \to 9} \frac{1}{\sqrt{x}+3} = \frac{1}{6}. \end{aligned}$$

We also have

$$\begin{aligned} \lim_{h \to 0} \frac{\sqrt{9+h}-3}{h} &= \lim_{h \to 0} \frac{\sqrt{9+h}-3}{h} \left(\frac{\sqrt{9+h}+3}{\sqrt{9+h}+3} \right) \\ &= \lim_{h \to 0} \frac{h}{h(\sqrt{9+h}+3)} \\ &= \lim_{h \to 0} \frac{1}{\sqrt{9+h}+3} = \frac{1}{6}. \end{aligned}$$

Either way, the answer is 1/6.

3. (20 points) Implicit differentiation and differentials. We want to understand the curve

$$y \ln(x) + xy^2 = 4$$

near the point $(1, 2)$.

a) Use implicit differentiation to compute $y'(1)$.

$$\begin{aligned} \frac{d}{dx}(y \ln(x) + xy^4 &= \frac{d}{dx} 4 \\ \frac{y}{x} + y^2 + y'(\ln(x) + 2xy) &= 0 \end{aligned}$$

$$
\begin{aligned}
6 + 4y' &= 0 \\
y' &= -3/2.
\end{aligned}
$$

b) Find the equation of the line tangent to the curve at $(1, 2)$.

Since $y - 2 = (-3/2)(x - 1)$, we have $y = 2 - \frac{3}{2}(x - 1)$.

c) Use a linear approximation (aka linearization, aka differentials) to find the approximate value of y when $x = 1.06$.

Plugging $x = 1.06$ into the equation of the tangent line gives $y = 2 - \frac{3}{2}(0.06) = 1.91$.

4. (20 points) Related rates. Sand is being dropped to form a sandpile. The sandpile always has the shape of a (point side up) cone with a circular base, and with a height equal to the radius. The *volume* of the sand pile is growing at a rate of 75π cubic feet/minute. We want to understand how fast the *radius* is changing. [Note: the volume of a cone with radius r and height h is $\pi r^2 h/3$.]

a) Draw a picture of the situation and define your variables.

This should be an upside-down cone whose height is h and whose base is a circle with radius r, and whose volume is V. Since $h = r$, we have $V = \pi r^2 h/3 = \pi r^3/3$. Taking derivatives with respect to time gives $dV/dt = \pi r^2 (dr/dt)$.

b) Compute the rate at which the radius is increasing when the radius equals 5 feet.

$$\frac{dr}{dt} = \frac{dV/dt}{\pi r^2} = \frac{75\pi}{25\pi} = 3$$

(feet per minute).

c) Compute the rate at which the radius is increasing when the radius equals 10 feet.

$$\frac{dr}{dt} = \frac{dV/dt}{\pi r^2} = \frac{75\pi}{100\pi} = 3/4$$

(feet per minute).