M408R Second Midterm Exam Solutions, October 24, 2014

1) This problem is a series of short questions, each worth 6 points (a–e) or 5 points (f and g).

a) Compute df/dx, where $f(x) = 1027e^{2x}$.

By the chain rule, $f'(x) = 2054e^{2x}$.

b) Compute dh/dx, where $h(x) = \ln(\cos(2x))$

By the chain rule, applied twice, $h'(x) = \frac{d(\cos(2x))/dx}{\cos(2x)} = \frac{-2\sin(2x)}{\cos(2x)} = -2\tan(2x)$.

c) Compute the derivative of $j(x) = \sin(x)e^{(x^2)}$.

This is the product rule: $j'(x) = 2x\sin(x)e^{(x^2)} + \cos(x)e^{(x^2)}$.

d) Compute the derivative of $k(x) = \frac{2x+1}{3x+7}$.

Quotient rule: $k'(x) = \frac{(3x+7)2-3(2x+1)}{(3x+7)^2} = \frac{11}{(3x+7)^2}$.

e) Find the partial derivative of $F(x,y) = \frac{e^{xy}}{y}$ with respect to x.

We treat y as a constant. The derivative of e^{xy} is ye^{xy}, so the partial derivative w.r.t. x of e^{xy}/y is e^{xy}.

f) Simplify $\sqrt{\ln(e^{25})}$ as much as possible.

Since $\ln(e^{25}) = 25$, the answer is 5.

g) Simplify $4^{\log_2(7)}$ as much as possible.

$(2^2)^{\log_2(7)} = 2^{2\log_2(7)} = (2^{\log_2(7)})^2 = 7^2 = 49$.

2) Consider the function $f(x) = \ln(x^2+1) - \ln(5)$.

a) Compute $f'(x)$.

By the chain rule, $f'(x) = 2x/(x^2+1)$. Note that $\ln(5)$ is a constant, whose derivative is zero.

b) Let L be the line tangent to the curve $y = f(x)$ at $x = 2$. Find the slope of L.

This is just $f'(2) = 4/5$.

c) Find the equation of L. (You can leave your answer in either point-slope or slope-intercept form.)

Since L goes through $(2, f(2)) = (2, 0)$ and has slope $4/5$, its equation is $y = (4/5)(x-2)$, or $y = \frac{4}{5}x - \frac{8}{5}$.

d) Use your answers to (b) and (c) to approximate $f(2.05)$. [Note: This is a problem about calculus, so just plugging $x = 2.05$ into your calculator and hitting the ln key is NOT worth any points. But it's not a bad way to check your answer.]

$f(2.05) \approx (4/5)(2.05-2) = (4/5)(.05) = 0.04$. You can also think of this as the microscope equation: $f(2.05) \approx f(2) + 0.05f'(2)$.

3) A colony of bacteria is undergoing exponential growth. That is, if $B(t)$ is the amount of bacteria (measured in grams) at time t (measured in hours), then

$$B(t) = Ce^{rt},$$

where C and r are unknown constants.

a) If there are 2 grams of bacteria at time $t = 0$, what is C?

Since $2 = B(0) = Ce^0 = C$, we must have $C = 2$.

b) 4 hours later, the mass of bacteria has grown to 2.83 grams. From this fact (and your answer to (a)), compute r.

Since $2.83 = B(4) = 2e^{4r}$, we have $e^{4r} = 2.83/2 = 1.415$, so $4r = \ln(1.415)$, or $r = \ln(1.415)/4 \approx 0.0867823828$. ($r$ comes in units of inverse hours, by the way.)

c) At some time later, there are 4 grams of bacteria. How fast (in grams/hour) is B changing at this later time?

Since $B'(t) = rB(t)$, $B'(t) = 4r \approx 0.34713$ grams/hour.

4) (Note: the following numbers are made up, but the problem of animal extinction is very real.) The population $E(t)$ of elephants in East Africa is decreasing at a rate proportional to the existing population. In the year 2000, there were 35,000 left. At that time, their number was decreasing at a rate of 1400 elephants/year.

a) Write down an initial value problem that governs the population of elephants. That is, write down i) a rate equation that captures all the information of the previous paragraph and ii) an initial condition. Your answers should be of the form
$E'(t) =$(some expression involving $E(t)$), $\qquad E$(some time) $=$ (some value).

Since the rate of change is proportional to E, we must have $E' = rE$. But $E' = -1400$ when $E = 35,000$, so $r = -1400/35000 = -0.04$. In other words, our rate equation is $E' = -0.04E$. If we let t be the number of years since 2000, then our initial condition is $E(0) = 35,000$. (If we had chosen t to be the actual date, then our initial condition would have been $E(2000) = 35,000$)

b) Write down the *solution* to this initial value problem. Your answer should be of the form $E(t) =$ (some explicit function of t).

The solution to $E' = rE$ is $E(t) = E(0)e^{rt}$. If t is the time since 2000, this means $E(t) = 35,000e^{-0.04t}$. (If t is the actual date, then $E(0)$ is the number of elephants 2014 years ago, which we get by solving $35000 = E(2000) = E(0)e^{-0.04(2000)}$, with the result that $E(0) = 35,000e^{80}$. We would then have $E(t) = 35,000e^{80-0.04t} = 35,000e^{-0.04(t-2000)}$.)

c) If this model is correct, how many elephants will be left in 2050? In 2100?

In 2050 there will only be $35000e^{-2} = 4737$ elephants (rounding to the nearest elephant), and in 2100 there will be $35000e^{-4} = 641$.

[In reality, poaching in East Africa was totally out of control for much of the late 20th century, with populations decreasing much *faster* than 4%/year. According to Wikipedia, from 1977 to 1989 the population dropped a whopping 74%. Fortunately, the population of elephants in southern Africa has been stable or even increasing, and recently the East African population has started to recover. Only time will tell what happens next.]