M408R Third Midterm Exam Solutions, November 21, 2014

1) Compute the following integrals, each worth 6 points. Note that they don't all use the same technique! Please indicate what technique you are using.

a) $\int \cos(z)(1-\sin^2(z))dz$.

Take $u = \sin(z)$, so $du = \cos(z)dz$. This turns the integral into $\int(1-u^2)du = u - u^3/3 + C = \cos(z) - \cos^3(z)/3 + C$.

b) $\int \frac{dw}{3w+1}$.

Take $u = 3w+1$, so $du = 3dw$, so $dw = du/3$, so we have the integral $\int \frac{du}{3u} = \ln(|u|)/3 + C = \ln(|3w+1|)/3 + C$.

c) $\int_0^1 \frac{2.4}{t^2+1}dt$.

The anti-derivative of $1/(1+t^2)$ is $\tan^{-1}(t)$, so we get $2.4\tan^{-1}(t)\Big|_0^1 = 2.4(\tan^{-1}(1) - \tan^{-1}(0))$. But $\tan^{-1}(1) = \pi/4$ and $\tan^{-1}(0) = 0$, so we get $2.4\pi/4 = 0.6\pi$.

d) $\int x^2 + \frac{1}{x^2}dx$.

It's easier if you write $1/x^2$ as x^{-2}. Then $\int(x^2+x^{-2})dx = x^3/3 - x^{-1} + C$.

e) $\int_1^e \frac{\ln(r)}{r}dr$.

Take $u = \ln(r)$, so that $du = dr/r$. We have $\int udu = u^2/2 + C = \ln(r)^2/2 + C$. For definite integrals we can pick any value of C that we like. Let's pick $C = 0$. Then our integral is $\ln(r)^2/2\Big|_1^e$. Since $\ln(e) = 1$ and $\ln(1) = 0$, this is $1/2$.

f) $\int(y^2+1)^2dy$.

u-substitution does NOT work here. Instead, just expand it out:

$$\int(y^2+1)^2dy = \int(y^4+2y^2+1)dy = y^5/5 + 2y^3/3 + y + C.$$

g) $\int_{-1}^3 |x|dx$. [Hint: Break it up as the sum of two definite integrals.]

Since $|x|$ is defined in pieces, it helps to work the integral in two pieces, one from -1 to 0 and one from 0 to 3. Each piece gives the area of a triangle, and the total area is $1^2/2 + 3^2/2 = 5$.

2) Using anti-derivatives. Let $a(t) = -9.8$.

a) Find an anti-derivative $v(t)$ of $a(t)$ such that $v(1) = 39.2$. (There is one and only one correct answer.)

$v(t) = -9.8t + C$. Since $39.2 = v(1) = C - 9.8$, we must have $C = 49$. So $v(t) = -9.8t + 49$.

b) Find an anti-derivative $z(t)$ of $v(t)$ such that $z(1) = 44.1$.

$z(t) = -4.9t^2 + 49t + \tilde{C}$. (I'm using a different symbol because this is a different C than in part (a)). But $44.1 = z(1) = -4.9 + 49 + \tilde{C} = 44.1 + \tilde{C}$, so $\tilde{C} = 0$. Thus $z(t) = -4.9t^2 + 49t$.

c) For what values of t is $z(t) = 0$?

Since $z(t) = -4.9t(t - 10)$, the values are $t = 0$ and $t = 10$.

d) A cannonball is shot into the air. At time $t = 1$ second, it is 44.1 meters above the ground and is rising at 39.2 meters/second. At all times, it is accelerating *downwards* at $9.8m/s^2$ because of gravity. At what time will it hit the ground? At what time was it fired?

This is **exactly** the same problem, only expressed in terms of physics instead of math. $a(t)$ is acceleration, $v(t)$ is velocity, and $z(t)$ is height. It will hit the ground at $t = 10$ and was fired at $t = 0$.

3) A snowstorm is expected to hit the town of Frostbite Falls tonight, starting at midnight. The rate at which snow falls is expected to be given by the function $r(t) = 5e^{-2t}$ inches/hour, where t is the number of hours since midnight. (E.g., 1:30 AM is $t = 1.5$, so at 1:30 AM the snow is coming down at a rate of $5e^{-3}$ inches/hour)

a) Write down a definite integral that gives the predicted total snowfall between midnight and 5AM. **Explain your reasoning!**

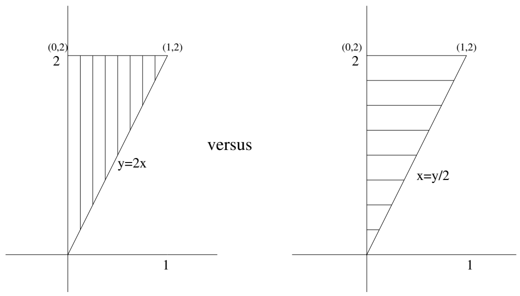
The amount of snow that falls in a time interval Δt is $r(t)\Delta t$. Adding these up and taking a limit gives $\int_0^5 r(t)dt = \int_0^5 5e^{-2t}dt$.

b) Now compute that integral. Give your answer first as an exact expression involving e, and then plug that expression into your calculator to get a numerical answer.

Let $u = -2t$. Then we have $\int(-5/2)e^u du = -(5/2)e^u + C = -(5/2)e^{-2t} + C$. Picking $C = 0$ we get the total to be $-(5/2)(e^{-10} - e^0) = 2.5(1 - e^{-10})$.

Numerically this works out to 2.49989 inches. (Since e^{-10} is practically zero, the answer is practically 2.5.)

4) Let T be a triangle in the x-y plane with vertices at the origin, at $(0,2)$, and at $(1,2)$. In other words, it is bounded by pieces of the y-axis, the line $y=2$, and the line $y=2x$ (aka $x=y/2$). We are going to compute the area of T in two different ways.



a) Write down an integral over x that gives the area of T. Explain where this integral comes from!

Cutting into slices of width $\Delta x = 1/N$, running from $y=2x$ to $y=2$, the height of each strip is $2-2x$, so we get $\sum(2-2x_i^*)\Delta x$. Taking a limit gives $\int_0^1 (2-2x)dx = 2x - x^2\Big|_0^1 = 1$.

Your answer in (a) should encode what happens when you cut T up into a bunch of vertical strips, add up the approximate area of each strip, and take a limit (as in the first figure). Instead, imagine cutting T up into N **horizontal** strips, all of the same height Δy, as in the second figure.

b) What is the width of each strip, as a function of y?

Since each horizontal strip runs from the line $x=0$ (i.e. the y axis) to the line $x=y/2$, it has width $y/2$.

c) What is the approximate area of each strip?

The strip is an approximate rectangle, with area $(y/2)\Delta y$ (or y/N, since $\Delta y = 2/N$).

d) Write down a Riemann sum (based on this approach) that gives the approximate area of T.

Adding all the answers from (c) gives $\sum_{i=1}^N \frac{y_i^*}{2}\Delta y$.

e) Now write down an integral that is the limit of this Riemann sum as we slice things finer and finer. [Voila! You've found a second way to compute areas by integration.]

The function we are integrating is $y/2$, and we start at $y = 0$ and work up to $y = 2$, so the answer is $\int_0^2 (y/2)dy$.

[FWIW, both $\int_0^1 (2 - 2x)dx$ and $\int_0^2 (y/2)dy$ work out to 1, which is the area of a triangle of base 1 and height 2.]