M408R Final Exam, December 13, 2016

1) Plug-and-chug grab bag (2 pages): Each part is worth 4 points.

a) Simplify $\sin^{-1}(\cos(\pi/8))$. Give an *exact* answer. (Plugging numbers into your calculator doesn't cut it.)

We want an angle whose sine is the cosine of $\pi/8$. That's the complement of $\pi/8$, namely $\frac{\pi}{2} - \frac{\pi}{8} = \frac{3\pi}{8}$.

b) Simplify: $e^{3\ln(3)} + \ln(e^{23}/e^{42})$.

$3^3 + (23 - 42) = 8$.

c) Compute $\dfrac{d}{dx}\sin(\ln(e^{5x} + \tan(x)))$.

By the chain rule (applied three times), the answer is

$$\cos(\ln(e^{5x} + \tan(x)))\frac{5e^{5x} + \sec^2(x)}{e^{5x} + \tan(x)}$$

d) Compute $f'(x)$ if $f(x) = \dfrac{\ln(x)}{x^2+1}$.

By the quotient rule, we get

$$\frac{\frac{x^2+1}{x} - 2x\ln(x)}{(x^2+1)^2}.$$

e) Compute $f(t)$ if $f'(t) = 6t^2 - 4t + 7$ and $f(1) = 13$.

Taking anti-derivatives, we get $f(t) = 2t^3 - 2t^2 + 7t + C$. Since $13 = f(1) = 7 + C$, we must have $C = 6$, so

$$f(t) = 2t^3 - 2t^2 + 7t + 6.$$

f) Find the acceleration at time $t = 2$ of a particle whose position is given by the function $x(t) = \cos^2(\pi t)$.

$v(t) = x'(t) = -2\pi\sin(\pi t)\cos(\pi t)$.
$a(t) = v'(t) = -2\pi^2(\cos^2(\pi t) - \sin^2(\pi t))$.
$a(2) = -4\pi^2(1^2 - 0^2) = -4\pi^2$.

g) Compute $\dfrac{d}{dx}\displaystyle\int_{2x}^{e^x} 2\cos(t^2+5)dt$.

Let $F(t)$ be the anti-derivative of $f(t) = 2\cos(t^2+5)$. We are taking the derivative of $F(e^x) - F(2x)$, which gives

$$e^x f(e^x) - 2f(2x) = 2e^x\cos(e^{2x}+5) - 4\cos(4x^2+5).$$

h) Suppose that $f(x) = \cos(\pi \ln(x))$. Compute $\displaystyle\int_1^e f'(x)dx$.

By the second fundamental theorem of calculus, this is $f(e) - f(1) = \cos(\pi) - \cos(0) = -2$.

i) Compute $\displaystyle\int_0^1 \frac{e^x}{(e^x+1)^2}dx$.

Do a u-substitution with $u = e^x + 1$ to get $\left.\frac{-1}{e^x+1}\right|_0^1 = \frac{1}{2} - \frac{1}{e+1}$.

j) Compute $\displaystyle\int 4x\cos(2x)dx$.

Integrate by parts with $u = x$, $dv = 4\cos(2x)dx$, $du = dx$, $v = 2\sin(2x)$ to convert this to

$$2x\sin(2x) - \int 2\sin(2x)dx = 2x\sin(2x) + \cos(2x) + C$$

2) A fishy business. (2 pages, 12 pts) When there is severe over-fishing, fish populations can no longer be modeled with exponential growth and decay. Instead, we have a differential equation in which the rate at which fish die is proportional to the poputation of fish, and the rate at which fish are born (or rather, hatched) is proportional to the square of the population. That is, the population $P(t)$ follows a differential equation of the form

$$P' = \alpha P^2 - \beta P,$$

where α and β are constants, where the first term represents the birth rate, and where the second term represents the death rate. (We are measuring time in years, by the way).

a) If the average lifespan of a fish is 10 years, what does β equal? Explain your reasoning.

If fish live an average of 10 years, then 1/10 of the fish will die each year, so $\beta = 1/10$. This is EXACTLY the same reasoning we used for the recovery coefficient in the SIR model.

b) Now suppose that $P' = 100$ when $P = 1000$. What is α?

If $100 = \alpha(1,000)^2 - 1,000/10$, then α works out to be $2 \times 10^{-4} = 0.0002$.

c) Using these values of α and β and the initial value $P(0) = 1000$, use Euler's method, with step size of 1/2, to approximate $P(1)$.

t	P	$P' = 2P^2/10^4 - P/10$	$P'\Delta t$
0	1000	100	50
0.5	1050	115.5	57.75
1.0	1107.75		

d) If the fish population were ever to drop below a certain threshold value, it would never recover. Compute this threshold.

The threshold is where $0 = P' = \alpha P^2 - \beta P = P(\alpha P - \beta)$, in other words where $P = \beta/\alpha = 0.1/0.0002 = 500$.

3. Tangent lines and microscopes. (10 pts) Let $f(x) = e^{(x^2-1)}(1 + 3\ln(x))$.

a) Compute $f'(1)$.

By the product rule, $f'(x) = 2xe^{x^2-1}(1 + 3\ln(x)) + \frac{3e^{x^2-1}}{x}$. Plugging in $x = 1$ gives $f'(1) = 2 + 3 = 5$.

b) Find the equation of the line tangent to the curve $y = f(x)$ at $(1, f(1))$.

Meanwhile, $f(1) = 1$. So our tangent line is $y - 1 = 5(x - 1)$, or $y = 1 + 5(x - 1)$ (or, if you insist, $y = 5x - 4$).

c) Find the approximate value of $f(1.002)$.

$f(1.002) \approx 1 + 5(1.002 - 1) = 1.01$.

4. Unicorn math (10 pts). Sadly, the population of unicorns in the Enchanted Forest is undergoing exponential decay. In the year 2010, there were 1000 unicorns. In 2015, there were only 800.

a) Find a formula for the number of unicorns as a function of time. (Since unicorns are magical creatures, their number does not have to be an integer, but can change continuously.)

There are two ways to do this, depending on whether you prefer to do your exponentials base e or base something else. Working base something-else, the simplest expression is $U(t) = 1000(0.8)^{t/5}$, where t is the number of years since 2010, and $U(t)$ is the unicorn population.

Working base e, we write instead $U(t) = 1000e^{-kt}$ and use the fact that $U(5) = 800$ to solve for k. The result is $k = -\ln(0.8)/5$, so $U(t) = 1000e^{t\ln(0.8)/5}$, which is the same thing as $1000(0.8)^{t/5}$.

b) How many unicorns will be left in 2030?

This is just plugging in $t = 20$ into the formula you found for $U(t)$. No matter which version you used, the answer is $1,000(0.8)^4 = 409.6$.

5. Riemann sums. (8 pts) The integral

$$\int_0^1 e^{-x^2/2} dx$$

cannot be computed by finding a formula for the anti-derivative. Instead, estimate its value by dividing the interval $[0,1]$ into four pieces and doing a Riemann sum. Be clear about whether you are using left endpoints, right endpoints, or some other scheme.

Let $f(x) = e^{-x^2/2}$. We are dividing the interval $[0,1]$ into 4 pieces, with $x_0 = 0$, $x_1 = 1/4$, $x_2 = 1/2$, $x_3 = 3/4$ and $x_4 = 1$, with spacing $\Delta x = 1/4$. We either want $\sum_{k=1}^4 f(x_k)\Delta x$ (right endpoints) or $\sum_{k=1}^4 f(x_{k-1})\Delta x$ (left endpoints). So we pull out our calculators and make a table of values of f:

x	$f(x)$
0	1
0.25	0.96923
0.5	0.88250
0.75	0.75484
1.0	0.60653

With left endpoints we get 0.90164 and with right endpoints we get 0.90327.

6. Present value. (10 pts) Annuities are contracts that are guaranteed to pay a certain amount of money per year for a certain length of time. Economists use the term "present value" to mean what you would have to pay today to buy such an annuity, and this depends on the prevailing interest rate r. If an annuity pays an A dollars per year for T years, then the present value is given by the integral

$$PV = \int_0^T Ae^{-rt} dt.$$

a) Compute this integral and simplify. Your answer should depend on the parameters A, r and T.

The anti-derivative of Ae^{-rt} is $-\frac{A}{r}e^{-rt}$. Plugging in at times 0 and T gives $\frac{A}{r}(1 - e^{-rT})$.

b) Now suppose that the annuity pays 1000 dollars/year, that it runs for 15 years, and the the prevailing interest rate is $r = 0.04$. What is the annuity worth?

Plugging in $A = 1,000$, $r = 0.04$ and $t = 15$ gives $25,000(1 - e^{0.6}) \approx 11280$. Since a future dollar is less valuable that a current dollar, getting \$15,000 spread out over 15 years is only work a little over \$11,000.

7. The whole is the sum of the parts. (10 points, 2 pages) An *annulus* (Latin for "ring") is the region between two concentric circles, as with a washer. We are going to use calculus to compute the area of an annulus with inner radius 1 and outer radius 2, as shown in the figure.

`(The figure shows an annulus of inner radius 1 and outer radius 2 divided into 4 concentric sub-annuli, each of thickness 1/4. It looks a lot like the logo for Target, only with the center bullseye missing.)`

We are going to break the annulus into N smaller pieces, all of the same thickness, as shown in the figure with $N = 4$. (I will call the smaller pieces "rings", and save the word "annulus" to mean the whole region.) We will estimate the area of each ring, add them up, take a limit as $N \to \infty$, and get an integral that we can then evaluate.

a) In terms of N, how thick is each ring?

$\Delta r = (2 - 1)/N = 1/N$.

b) Approximate the area of the k-th ring as its circumference times its thickness. **Indicate whether you are using the circumference of the inner edge of the ring, the outer edge, or something else.** Leave your answer as an explicit function of k and N. (E.g. something like $5\pi^2 \sin(3 + \frac{k}{N})$, although that is NOT the correct answer!)

The radius (which I'm calling r) goes from $a = 1$ to $b = 2$ in N steps of size $\Delta r = 1/N$. The k-th ring has inner radius $r_{k-1} = 1 + (k - 1)/N$ and outer radius $r_k = 1 + k/N$. If you use the outer edge, the area is approximately $2\pi r_k \Delta r = 2\pi(1 + k/N)/N$. If you use the inner edge, it would be $2\pi r_{k-1} \Delta r = 2\pi(1 + (k - 1)/N)/N$.

c) Now express the total area of the annulus as a Riemann sum.

Using outer radii (aka right endpoints) we have

$$\sum_{k=1}^{N} 2\pi r_k \Delta r = \sum_{k=1}^{N} 2\pi (1 + \frac{k}{N}) \frac{1}{N}.$$

To get the formula for inner radii, just replace all the k's with $(k-1)$'s.

d) Write down an integral that gives the limit of your Riemann sum as $N \to \infty$.

By definition, the limit of that sum as $N \to \infty$ is $\int_1^2 2\pi r dr$.

e) Evaluate this integral. [Hint: You can check your answer by comparing it to the difference between the area of a circle of radius 2 and the area of a circle of radius 1.] $\int_1^2 2\pi r dr = \pi r^2 \Big|_1^2 = 3\pi$.