M408R Final Exam, December 13, 2016

1) Plug-and-chug grab bag (2 pages): Each part is worth 4 points.

a) Simplify $\sin^{-1}(\cos(\pi/8))$. Give an *exact* answer. (Plugging numbers into your calculator doesn't cut it.)

b) Simplify: $e^{3\ln(3)} + \ln(e^{23}/e^{42})$.

c) Compute $\dfrac{d}{dx}\sin(\ln(e^{5x} + \tan(x)))$.

d) Compute $f'(x)$ if $f(x) = \dfrac{\ln(x)}{x^2+1}$.

e) Compute $f(t)$ if $f'(t) = 6t^2 - 4t + 7$ and $f(1) = 13$.

f) Find the acceleration at time $t = 2$ of a particle whose position is given by the function $x(t) = \cos^2(\pi t)$.

g) Compute $\dfrac{d}{dx}\displaystyle\int_{2x}^{e^x} 2\cos(t^2+5)dt$.

h) Suppose that $f(x) = \cos(\pi\ln(x))$. Compute $\displaystyle\int_1^e f'(x)dx$.

i) Compute $\displaystyle\int_0^1 \frac{e^x}{(e^x+1)^2}dx$.

j) Compute $\displaystyle\int 4x\cos(2x)dx$.

2) A fishy business. (2 pages, 12 pts) When there is severe over-fishing, fish populations can no longer be modeled with exponential growth and decay. Instead, we have a differential equation in which the rate at which fish die is proportional to the poputation of fish, and the rate at which fish are born (or rather, hatched) is proportional to the square of the population. That is, the population $P(t)$ follows a differential equation of the form

$$P' = \alpha P^2 - \beta P,$$

where α and β are constants, where the first term represents the birth rate, and where the second term represents the death rate. (We are measuring time in years, by the way).

a) If the average lifespan of a fish is 10 years, what does β equal? Explain your reasoning.

b) Now suppose that $P' = 100$ when $P = 1000$. What is α?

c) Using these values of α and β and the initial value $P(0) = 1000$, use Euler's method, with step size of $1/2$, to approximate $P(1)$.

d) If the fish population were ever to drop below a certain threshold value, it would never recover. Compute this threshold.

3. Tangent lines and microscopes. (10 pts) Let $f(x) = e^{(x^2-1)}(1 + 3\ln(x))$.

a) Compute $f'(1)$.

b) Find the equation of the line tangent to the curve $y = f(x)$ at $(1, f(1))$.

c) Find the approximate value of $f(1.002)$.

4. Unicorn math (10 pts). Sadly, the population of unicorns in the Enchanted Forest is undergoing exponential decay. In the year 2010, there were 1000 unicorns. In 2015, there were only 800.

a) Find a formula for the number of unicorns as a function of time. (Since unicorns are magical creatures, their number does not have to be an integer, but can change continuously.)

b) How many unicorns will be left in 2030?

5. Riemann sums. (8 pts) The integral

$$\int_0^1 e^{-x^2/2} dx$$

cannot be computed by finding a formula for the anti-derivative. Instead, estimate its value by dividing the interval $[0, 1]$ into four pieces and doing a Riemann sum. Be clear about whether you are using left endpoints, right endpoints, or some other scheme.

6. Present value. (10 pts) Annuities are contracts that are guaranteed to pay a certain amount of money per year for a certain length of time. Economists use the term "present value" to mean what you would have to pay today to buy such an annuity, and this depends on the prevailing interest rate r. If an annuity pays an A dollars per year for T years, then the present value is given by the integral

$$PV = \int_0^T Ae^{-rt} dt.$$

a) Compute this integral and simplify. Your answer should depend on the parameters A, r and T.

b) Now suppose that the annuity pays 1000 dollars/year, that it runs for 15 years, and the the prevailing interest rate is $r = 0.04$. What is the annuity worth?

7. The whole is the sum of the parts. (10 points, 2 pages) An *annulus* (Latin for "ring") is the region between two concentric circles, as with a washer. We are going to use calculus to compute the area of an annulus with inner radius 1 and outer radius 2, as shown in the figure.

`(The figure shows an annulus of inner radius 1 and outer radius 2 divided into 4 concentric sub-annuli, each of thickness 1/4. It looks a lot like the logo for Target, only with the center bullseye missing.)`

We are going to break the annulus into N smaller pieces, all of the same thickness, as shown in the figure with $N = 4$. (I will call the smaller pieces "rings", and save the word "annulus" to mean the whole region.) We will estimate the area of each ring, add them up, take a limit as $N \to \infty$, and get an integral that we can then evaluate.

a) In terms of N, how thick is each ring?

b) Approximate the area of the k-th ring as its circumference times its thickness. **Indicate whether you are using the circumference of the inner edge of the ring, the outer edge, or something else.** Leave your answer as an explicit function of k and N. (E.g. something like $5\pi^2 \sin(3 + \frac{k}{N})$, although that is NOT the correct answer!)

c) Now express the total area of the annulus as a Riemann sum.

d) Write down an integral that gives the limit of your Riemann sum as $N \to \infty$.

e) Evaluate this integral. [Hint: You can check your answer by comparing it to the difference between the area of a circle of radius 2 and the area of a circle of radius 1.]