M408N First Midterm Exam Solutions, September 21, 2016

1) The SIR model. Suppose that an epidemic is modeled by the SIR model:

$$\begin{aligned} S' &= -aSI \\ I' &= aSI - bI \\ R' &= bI \end{aligned}$$

where the transmission coefficient is $a = 0.00025$ and the recovery coefficient is $b = 0.1$. (Here we are measuring S, I and R in people and time in days.) The initial conditions on day 0 are:

$$S(0) = 1000; \qquad I(0) = 20; \qquad R(0) = 0.$$

a) Compute $S'(0)$, $I'(0)$ and $R'(0)$.

$$\begin{aligned} S'(0) &= -0.00025(1000)(20) &&= -5 \\ I'(0) &= .00025(1000)(20) - .1(20) &&= 3 \\ R'(0) &= .1(20) &&= 2. \end{aligned}$$

b) Estimate $S(1)$, $I(1)$ and $R(1)$.

$$\begin{aligned} S(1) &\approx S(0) + 1(S'(0)) = 1000 - 5 = 995 \\ I(1) &\approx I(0) + 1(I'(0)) = 10 + 3 = 23 \\ R(1) &\approx R(0) + 1(R'(0)) = 0 + 2 = 2. \end{aligned}$$

c) How many more people need to get sick (beyond the initial 20) before the epidemic starts to abate? [Hint: the buzzword is "threshold"]

The epidemic hits its peak when $0 = I' = I(aS - b)$, in other words when S hits the threshold of $b/a = 400$. Since S starts off at 1000, this means 600 more people have to get sick before the epidemic starts to abate. (And still more people will get sick in the later stages of the epidemic. This is a BIG public health problem.)

d) Suppose that an effective public health program reduced the transmission coefficient to 0.000125. In those circumstances, how many people would need to get sick before the epidemic started to abate.

This brings the threshold up to 800, so things will start to settle down after 200 people get sick. A little prevention went a long way!

2) Rate equations. Consider the rate equation

$$y'(t) = t + 5y - y^2,$$

with initial value $y(1) = 3$.

a) Use Euler's method, with step size $\Delta t = 0.2$, to approximate $y(1.2)$.

t	y	y'	$y'\Delta t$
1	3	7	1.4
1.2	4.4		

so $y(1.2) \approx 4.4$.

b) Use Euler's method, with step size $\Delta t = 0.1$, to approximate $y(1.2)$.

t	y	y'	$y'\Delta t$
1	3	7	0.7
1.1	3.7	5.91	.591
1.2	4.291		

so $y(1.2) \approx 4.291$. This is a more accurate approximation than in part (a).

3) Suppose that $f(x)$ is a differentiable function, and that

$$f(0.99) = 0.36418, \qquad f(1) = 0.36788, \qquad f(1.01) = 0.37154.$$

a) Estimate the value of $f'(1)$.

If you use forward differences, this is $(f(1.01) - f(1))/.01 = .366$
If you use centered differences, this is $(f(1.01) - f(0.99))/.02 = .368$
If you use backward differences, this is $(f(1) - f(0.99))/.01 = .370$
All of those were considered correct answers.

b) To within the accuracy of your approximation in part (1), find the equation of the line tangent to the graph $y = f(x)$ at $(1, f(1))$.

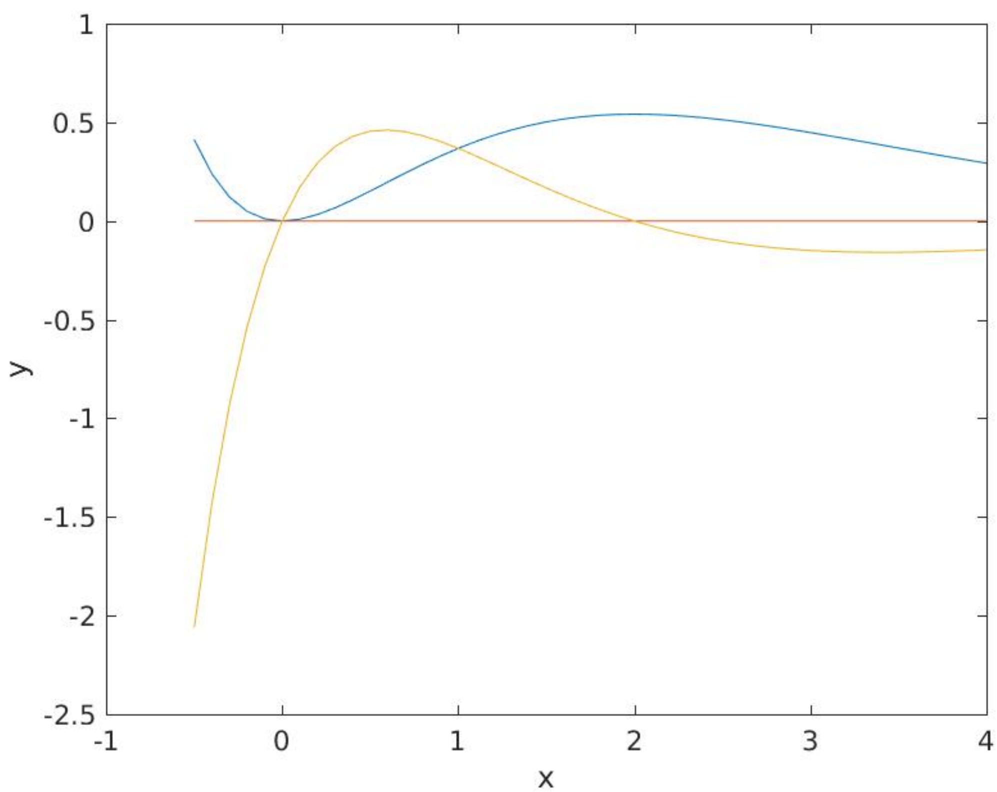
If you used centered differences, this would be $y = f(1) + f'(1)(x - 1) = .36788 + 0.368(x - 1)$. For forward of backward differences, just replace .368 with .366 or .370. Note that this is by far the most useful form of the equation. The form $y = .368x - 0.00012$, while correct, is much harder to work with.

c) Estimate the value of $f(1.062)$.

Just plug in $x = 1.062$ into the last equation to get $f(1.062) \approx .36788 + 0.368(0.062) = .390696$ (That's with centered differences. Forward gives .390572 and backward gives .390820)

4) Here is the graph $y = f(x)$ of a mystery function f between $x = -0.5$ and $x = 4$. (Generated in MATLAB, by the way) On the SAME set of axes, sketch the graph of $f'(x)$. The graph does not have to be precise, but you should clearly indicate:

- Where f' is positive.
- Where f' is negative.
- Where f' is zero.
- Where f' is relatively big, and where it's small.



The graph has been altered to include a plot of $f'(x)$. The graph of $f(x)$ is in teal, and that of $f'(x)$ is in yellow. By the way, $f(x) = x^2e^{-x}$ and $f'(x) = (2x - x^2)e^{-x}$.

A remarkably large fraction of the class drew very accurate sketches. Way to go!