M408R Second Midterm Exam, October 17, 2016

1. (40 points, 2 pages) Compute the following derivatives.

a) $h'(t)$, where $h(t) = (3t^2 + 5t + 1)^{35}$.

By the chain rule, $h'(t) = 35(3t^2 + 5t + 1)^{34}(6t + 5)$.

b) $f'(x)$, where $f(x) = e^{5x} \cos(x)$.

By the product rule, $f'(x) = 5e^{5x} \cos(x) - e^{5x} \sin(x)$.

c) dg/dt, where $g(t) = \frac{3t+5}{2t^2+7}$.

By the quotient rule, this is

$$\frac{dg}{dt} = \frac{(2t^2 + 7)3 - (3t + 5)(4t)}{(2t^2 + 7)^2} = \frac{-6t^2 - 20t + 21}{(2t^2 + 7)^2}.$$

d) $F'(5)$, where $F(x) = H(x^2 - 5)$ and $H'(20) = -3$.

$F'(x) = 2xH'(x^2 - 5)$, so $F'(5) = 10H'(20) = -30$.

e) The derivative of $x \ln(x) - x$ with respect to x. Simplify as much as possible.

By the product rule, this is $x\frac{1}{x} + \ln(x) - 1 = \ln(x)$.

f) dy/dx, where $y = u \ln(u) - u$ and $u = e^x + \tan^{-1}(x)$. Express your answer as a function of x only.

By the chain rule and the previous problem,

$$\frac{dy}{dx} = \ln(u)\frac{du}{dx} = \ln\left(e^x + \tan^{-1}(x)\right)\left(e^x + \frac{1}{1 + x^2}\right).$$

g) The partial derivative of $\sin(xy^2 + 5)$ with respect to x.

By the chain rule, this is $y^2 \cos(xy^2 + 5)$.

h) The partial derivative of $\sin(xy^2 + 5)$ with respect to y.

By the chain rule, this is $2xy \cos(xy^2 + 5)$.

2. (20 points) Derivative fundamentals. In this problem we will compute a derivative of the function $f(x) = 1/x$ **from the definition**. Just plugging into a convenient formula for derivatives will not get you **any points whatsoever**.

a) What is the definition of $f'(5)$ as a limit? (There are multiple correct answers. One is enough.)

$$f'(5) = \lim_{x\to 5} \frac{(1/x) - (1/5)}{x - 5} = \lim_{h\to 0} \frac{1/(5 + h) - (1/5)}{h} = \lim_{h\to 0} \frac{1/(5 + h) - 1/(5 - h)}{2h}.$$

Any of these expressions is worth full credit.

b) Evaluate this limit to compute $f'(5)$.

Since

$$\frac{\frac{1}{x}-\frac{1}{5}}{x-5} = \frac{\frac{5}{5x}-\frac{x}{5x}}{x-5} = \frac{-1}{5x},$$

the first limit is $-1/25$. Similarly, the second limit works out to

$$\lim_{h\to 0} \frac{-1}{5(5+h)} = \frac{-1}{25}.$$

3. (20 points) Population. Since the Civil War, the population of the Austin metropolitan area has grown at a steady rate of 3.5%/year. In the year 2000, the population was (approximately) 1,000,000. Assuming this growth pattern continues,

a) What will the population of Austin be in the year $2000+t$? (That is, we are measuring time in years since 2000.)

This is exactly the setting for exponential growth. $P(t) = 1,000,000e^{.035t}$.

b) What will the population be in 2060?

$P(60) = 1,000,000e^{60(.035)} = 1,000,000e^{2.1} = 8,166,170$. You can get (approximately) the same result from the law of 70. The doubling time is 20 years, so the population will double three times in 60 years, from one million to 2 million to 4 million to 8 million.

c) When will the population hit 5,000,000?

$$\begin{aligned} 5,000,000 &= 1,000,000e^{.035t} \\ 5 &= e^{.035t} \\ \ln(5) &= .035t \\ t &= \ln(5)/.035 = 45.98 \approx 46. \end{aligned}$$

The population will hit 5 million in 2046.

4. (20 points, 2 pages) Modeling with differential (aka rate) equations. An eccentric donor is setting up the Fund for the Assistance of Calculus Teachers (FACT!) to provide comfort food for UT instructors who are grading calculus exams. (Alas, this fund is actually fictitious.) The donor will continuously supply money to the fund at a rate of \$100/month. Meanwhile, money in the fund is to be withdrawn continuously at a rate of 10%/month to pay for pizza, ice cream and adult beverages. (For instance, if there were \$200 in the fund, then at that point in time money would be coming in at a rate of \$100/month and going out at a rate of \$20/month.) Let $y(t)$ be the amount of money in the fund after t months.

a) Write down a rate equation that describes what is happening. That is, what is y' in terms of y? (I am not asking you to *solve* the rate equation, just to formulate it.)

Since money (measured in dollars) is coming in at 100/month and going out at $0.1y$/month, we have

$$y' = 100 - (0.1)y = 100 - y/10.$$

b) For what values of y is y increasing? For what values of y is y decreasing? Is there some sort of threshold?

When $y < 1000$, $y/10$ is less than 100, so $y' > 0$ and y is increasing. When $y > 1000$, $y/10 > 100$, so $y' < 0$ and y is decreasing. The threshold in this problem is 1000.

c) Suppose that $y(0) = 100$. Based on your answer to part (b), sketch a graph of y as a function of t. It does NOT have to be precise, but you should get the qualitative behavior right. (E.g. if y goes up for a while, then drops, and then levels off at $y = 200$, you should draw a curve that goes up, then down, and then levels off at $y = 200$.)

Since 100 is (way) below the threshold, y will increase with time, leveling off as it approaches the threshold from below. For a more precise picture, you should ask MATLAB to plot the exact solution, which works out to be $y(t) = 1000 - 900e^{-t/10}$.