

M408R Third Midterm Exam Solutions, November 21, 2016

1. (30 points, 2 pages) Compute the following integrals and anti-derivatives.

a)  $\int_1^2 \frac{x^2 + 3x + 2}{x} dx.$

Since  $\frac{x^2+3x+2}{x} = x + 3 + \frac{2}{x}$ , we get  $\frac{x^2}{2} + 3x + 2 \ln(x) \Big|_1^2 = \frac{9}{2} + 2 \ln(2).$

b)  $\int \frac{6x + 3 \cos(x)}{x^2 + \sin(x)} dx.$

Setting  $u = x^2 + \sin(x)$ , this is  $\int \frac{3du}{u} = 3 \ln(|u|) + C = 3 \ln(|x^2 + \sin(x)|) + C.$

c)  $\int_0^1 6(4x - 1)^2 dx$

This is  $\frac{(4x-1)^3}{2} \Big|_0^1 = \frac{3^3 - (-1)^3}{3} = \frac{28}{3}.$

d)  $\int \frac{dx}{4 + x^2}$

I *told* you to work this one at home! First rewrite the denominator as  $4(1 + \frac{x^2}{4})$  and then do a  $u$ -substitution with  $u = x/2$ . This gives

$$\int \frac{\frac{du}{2}}{2(1 + u^2)} = \frac{1}{2} \tan^{-1}(u) + C = \frac{1}{2} \tan^{-1}(x/2) + C.$$

e)  $\int_0^6 e^{-t/3} dt$

You can do a  $u$ -substitution with  $u = -t/3$  to get  $-3e^{-t/3} \Big|_0^6 = 3(1 - e^{-2}).$

2. (20 points) Sketching accumulation functions. The following is the graph  $y = f(x)$  of a function  $f(x)$ . Let  $A(X) = \int_0^X f(x) dx$  be the accumulation function.

[The sketch shows a function that is positive and decreasing for  $x < -2$ , is zero at  $x = -2$ , hits a minimum at  $x = -1$ , increases to zero at  $x = 0$ , hits a local maximum at  $x = 1$ , drops to zero at  $x = 2$ , and continues down from there. This is similar in shape to the graph  $y = 3x - x^3$ , except that the roots are at  $-2, 0$  and  $2$  instead of  $\pm\sqrt{3}$  and  $0$ .]

a) For what values of  $X$  is  $A(X)$  increasing? Decreasing?

Since  $A'(X) = f(X)$  by FTC1,  $A(X)$  is increasing when  $f(X)$  is positive, namely when  $X < -2$  or  $0 < X < 2$ , and is decreasing when  $f(X)$  is negative, namely when  $-2 < X < 0$  or  $X > 2$ .

b) For what values of  $X$  is the graph of  $A(X)$  concave up? Concave down?

Since  $A''(X) = f'(X)$ , the graph is concave up when  $f(X)$  is increasing, namely when  $-1 < X < 1$ , and is concave down for  $X < -1$  and  $X > 1$ .

c) Sketch the graph of  $A(X)$ . It doesn't have to be precise, of course, but it should have the right value at  $X = 0$ , should be going upwards and downwards at the right places, and should curve upwards and downwards at the right places.

The graph starts out negative, crossing the axis somewhere to the left of  $X = -2$ , then increasing to a local maximum at  $X = -2$ , and curving downwards until  $X = -1$ . From  $-1$  to  $1$  it is U-shaped, with a local minimum at  $(0, 0)$ . (Remember that  $A(0) = 0$ !) From  $1$  to  $2$  it levels out, reaching a local maximum at  $X = 2$ , then curves downwards and eventually crosses the  $X$ -axis.

3. (30 points) We are going to use integration to compute the volume of the unit ball. This ball is obtained by rotating the unit disk  $x^2 + y^2 \leq 1$  in the  $x$ - $y$  plane around the  $x$ -axis. As with the cone problem that we worked in class, we are going to slice the sphere into  $N$  more-or-less cylindrical slices, estimate the volume of each slice, add them up, and take a limit.

a) How thick is each slice (in terms of  $N$ )?

Since we are going from  $x = -1$  to  $x = 1$ , each slice has thickness  $\Delta x = 2/N$ .

b) If we cut the sphere at a particular value of  $x$ , we get a circle. What is the cross-sectional area of this circle, as a function of  $x$ ?

Note that the equation of the unit circle is  $x^2 + y^2 = 1$ , or equivalently  $y = \pm\sqrt{1 - x^2}$ . This makes the radius of our cross-section  $\sqrt{1 - x^2}$ , so the area is  $\pi r^2 = \pi(1 - x^2)$ .

c) The  $i$ -th slice corresponds to an interval  $[x_{i-1}, x_i]$ . What is the approximate volume of this slice? (You may use left endpoints, right endpoints, midpoints, or whatever representative point you wish.)

The volume is  $\pi(1 - (x_i^*)^2)\Delta x$ . If we use left endpoints, this becomes

$\pi \left( 1 - \left( -1 + \frac{2(i-1)}{N} \right)^2 \right) \frac{2}{N}$ , while if we use right endpoints we have  $i$  instead of  $i - 1$ .

d) Express the approximate total volume of the sphere as a Riemann sum. Be as explicit as possible.

This is just the sum over  $i$  of your answer to part (c), namely

$$\sum_{i=1}^N \pi \left( 1 - \left( -1 + \frac{2(i-1)}{N} \right)^2 \right) \frac{2}{N} \text{ or } \sum_{i=1}^N \pi \left( 1 - \left( -1 + \frac{2i}{N} \right)^2 \right) \frac{2}{N}.$$

e) Write down a definite integral, which is the limit of this Riemann sum, that gives the exact volume of the sphere.

$$\int_{-1}^1 \pi(1 - x^2)dx.$$

f) Evaluate this integral with the help of the Fundamental Theorem of Calculus.

$$\text{This is } \pi \left( x - \frac{x^3}{3} \right) \Big|_{-1}^1 = \frac{4\pi}{3}.$$

4. (20 points) Using antiderivatives. A particle is moving with acceleration  $a(t) = 12t^2 - 24t + 8$ , where  $t$  is measured in seconds and  $a$  is measured in feet/second<sup>2</sup>. At time  $t = 1$  it is moving with velocity  $v(1) = 0$  and has position  $s(1) = 5$ .

a) Find the velocity as a function of time. At what times is the particle moving forward? At what times is it moving backwards? At which instants is it (momentarily) not moving?

Since  $v(t)$  is the antiderivative of  $a(t)$ , we have  $v(t) = 4t^3 - 12t^2 + 8t + C_1$ . Setting  $0 = v(1) = 4 - 12 + 8 + C_1$ , we get  $C_1 = 0$ , so

$$v(t) = 4t^3 - 12t^2 + 8t = 4t(t - 1)(t - 2).$$

This is positive when  $0 < t < 1$  or when  $t > 2$ , and is negative when  $t < 0$  or  $1 < t < 2$ , and is zero at  $t = 0, 1$ , or  $2$ . That is, the particle moves backwards until  $t = 0$ , stops at  $t = 0$ , moves forwards until it stops again at  $t = 1$ , then moves backwards until it stops yet again at  $t = 2$ , and then moves forwards.

b) Find the position as a function of time.

The position  $s(t)$  is the antiderivative of velocity, so  $s(t) = t^4 - 4t^3 + 4t^2 + C_2$ . Setting  $5 = s(1) = 1 - 4 + 4 + C_2$  gives  $C_2 = 4$ , so

$$s(t) = t^4 - 4t^3 + 4t^2 + 4.$$