M403K Final Exam Solutions
May 13, 2002

Problem 1. Graphing

Consider the function $f(x) = x^4 - 2x^2$.

a) Find the partition points of f and make a sign chart for f.

$f(x) = x^2(x^2 - 2)$, so the partition points are 0, $\sqrt{2}$ and $-\sqrt{2}$. The function is positive for $x < -\sqrt{2}$ and for $x > \sqrt{2}$. It is negative for $-\sqrt{2} < x < 0$ and for $0 < x < \sqrt{2}$.

b) Find the critical points of f and make a sign chart for f'.

$f'(x) = 4x^3 - 4x = 4x(x^2 - 1)$, so the critical points are -1, 0 and 1. f' is negative for $x < -1$, positive for $-1 < x < 0$, negative for $0 < x < 1$ and positive for $x > 1$.

c) Find the inflection points of f and make a sign chart for f''.

$f''(x) = 4(3x^2 - 1)$, so the inflection points are at $\pm 1/\sqrt{3}$. f'' is positive for $x < -1/\sqrt{3}$ and for $x > 1/\sqrt{3}$, and negative for $-1/\sqrt{3} < x < 1/\sqrt{3}$.

d) On the back of this page, sketch the curve $y = f(x)$. Mark all important points CLEARLY.

I can't draw this on screen, but the graph looks like a "Mexican hat". There are local minima at $(-1, -1)$ and $(1, -1)$, and a local maximum at $(0, 0)$. The graph is positive, decreasing and curving up for $x < -\sqrt{2}$, then negative, decreasing and curving up for $-\sqrt{2} < x < -1$, hitting a local minimum at (-1, -1). It is then negative, increasing, and curving up for $-1 < x < -1/\sqrt{3}$, and is negative, increasing, and curving down for $-1/\sqrt{3} < x < 0$, hitting a local maximum at (0,0). The remainder of the graph is a mirror image of the first half, since $f(-x) = f(x)$.

Problem 2. Max-min

Consider the function $f(x) = e^{2x} - e^x$.

a) Find all critical points of this function. For each one, say whether it is a local maximum, a local minimum, or neither.

$f'(x) = 2e^{2x} - e^x$. Setting this equal to zero we have

$$2e^{2x} = e^x.$$

But $e^{2x} = e^{x+x} = e^x e^x$, so dividing our equation by e^x gives

$$2e^x = 1,$$

so $e^x = 1/2$, so $x = \ln(1/2) = -\ln(2) \approx -0.7$. This is the ONLY critical point. Now $e^{\ln(1/2)} = 1/2$, and $e^{2\ln(1/2)} = (1/2)^2 = 1/4$, so $f(\ln(1/2)) = (1/4) - (1/2) = -1/4$. Now $f''(x) = 4e^{2x} - e^x$, so $f''(\ln(1/2)) = 4/4 - 1/2 = 1/2 > 0$, so $x = \ln(1/2)$ is a local minimum.

b) Find the global maximum and minimum of $f(x)$ in the interval $[-5, 5]$.

The candidates are the critical points and the endpoints. $f(5) = e^{10} - e^5$ is a huge positive number, and $f(-5) = e^{-10} - e^{-5}$ is a tiny negative number, so the maximum is at $x = 5$ and the minimum is at $x = -\ln(2)$.

Problem 3. Marginal analysis

The demand x for widgets is related to the price p by the demand equation $x = 3000 - 100p$. The cost function is $C(x) = 500 + 20x - x^2/200$.

a) Find the marginal cost, the marginal revenue, and the marginal profit at a production level of $x = 1200$.

Solving for price in terms of demand gives $p = 30 - x/100$, so $R(x) = xp = 30x - x^2/100$. Since $C(x) = 500 + 20x - x^2/200$, we have a profit $P(x) = R(x) - C(x) = 10x - (x^2/200) - 500$. Taking derivatives we get our marginal quantities:

$$R'(x) = 30 - x/50, \qquad C'(x) = 2- = x/100, \qquad P'(x) = 10 - x/100.$$

Finally, plugging in $x = 1200$ gives $R'(1200) = 6$, $C'(1200) = 8$ and $P'(1200) = -2$. In other words, each additional widget costs us \$ 8, and only brings us \$ 6 in additional revenue, and so decreases our profit by \$ 2.

b) What is the production level that maximizes revenue? What production level maximizes profit?

To maximize revenue, set $R' = 0$. This gives $x = 1500$.

To maximize profit, set $P' = 0$. This gives $x = 1000$.

Problem 4. Exponential growth

An investor invests \$1000 at 7% interest, compounded continuously.

a) How much money will he have in 20 years? Express your answer as an exact expression (e.g. something like \$ $500e^{4.2}$ – no, that's not the right answer), and then approximate it numerically (e.g., \$16,000).

$Pe^{rt} = 1000e^{0.07t} = 1000e^{1.4}$. Now $\ln(2) \approx 0.7$ (the Law of 70, remember?), so $1.4 \approx 2\ln(2)$ and $e^{1.4} \approx e^{2\ln(2)} = 2^2 = 4$. Thus $\$1000e^{1.4} \approx \$4000$.

b) When will there be \$10,000 in his account? Express your answer as an exact expression. You do NOT need to approximate it numerically.

$10{,}000 = 1000e^{rt}$, so $10 = e^{rt}$, so $\ln(10) = rt$, so $t = \ln(10)/r = \ln(10)/0.07$. (This is a little under 33 years).

Problem 5. Rates of change

A quantity y is changing at a rate

$$\frac{dy}{dx} = 2e^x + \frac{3}{x+1} - 6x^2$$

When $x = 0$, $y = 5$. What does y equal when $x = 2$?

$$y = \int (2e^x + \frac{3}{x+1} - 6x^2)dx = 2e^x + 3\ln(x+1) - 2x^3 + C.$$

Plugging in $y(0) = 5$ we have $5 = 2e^0 + 3\ln(1) - 2(0)^3 + C = 2 + C$, so $C = 3$. Thus:

$$y(2) = 2e^2 + 3\ln(3) - 16 + 3 = 2e^3 + 3\ln(3) - 13.$$

Problem 6. Volume

A silo-shaped region is obtained by taking the region between the curve $y = 4 - x^2$ and the x-axis and rotating it about the y-axis. [See figure]. We compute the volume of this region by slicing it into a stack of disks.

Note: This is a straightforward application of the "slice and dice principle", but doesn't directly correspond to any formulas in the book. I'm quite disappointed that only 4 or 5 people in the entire class attempted this problem.

a) Find the (approximate) radius and (approximate) volume of a disk at height y and thickness Δy.

The radius of the disk is just the value of x on the curve. Since $y = 4 - x^2$, we have $x^2 = 4 - y$, so our radius is $r = \sqrt{4-y}$. The area is $\pi r^2 = \pi(4-y)$, and the volume is (area $\times$ thickness) $= \pi(4-y)\Delta y$.

b) By summing the answer to (a) and taking a limit, express the volume of the silo as a definite integral.

The sum of volumes is of the form $\sum f(y)\Delta y$, where $f(y) = \pi(4-y)$, and y ranges from 0 to 4. Taking the limit as the number of slices goes to infinity gives the definite integral $\int_0^4 \pi(4-y)dy$.

c) Evaluate this integral to get the total volume.

The integral evaluates to $\pi(4y - y^2/2)|_0^2 = 8\pi$.

Part II:

Evaluate the following. The limits and integrals should be simplified as much as possible, but you don't have to simplify the derivatives:

a) $\lim_{x\to 3} \frac{e^x - e^3}{x-3} = \lim_{x\to 3}(e^x/1) = e^3$

b) $\int_0^1 x(x^2+1)^3 dx$. Set $u = x^2+1$.

$$\int_0^1 x(x^2+1)^3 dx = \int_{x=0}^1 \frac{1}{2}u^3 du = u^4/8|_{x=0}^1 = \frac{(x^2+1)^4}{8}|_0^1 = \frac{2^4-1^4}{8} = 15/8.$$

c) $\int e^{2x} + 3x + 4dx$

$\frac{e^{2x}}{2} + \frac{3x^2}{2} + 4x + C.$

d) $D_x(x^2e^{-x}) = 2xe^{-x} - x^2e^{-x}$.

e) $f'(x)$, where $f(x) = [\ln(2x+1)]^3$. Apply the chain rule twice to get $6(\ln(2x+1))^2/(2x+1)$

f) $\int \frac{e^x}{1+e^x}dx = \ln(1+e^x)+C$. This comes from the substitution $u = 1+e^x$.

g) dy/dx, where $y = \frac{e^x}{\ln(x)}$ is $\frac{\ln(x)e^x - e^x/x}{(\ln(x))^2}$.

h) $\lim_{x\to\infty} \frac{3x^3-2x^2+7}{2x^3+1} = 3/2$. (Either apply L'Hopital 3 times, or divide top and bottom by x^3 and take a limit).

i) $\int_3^8 \sqrt{x+1}dx = (2/3)(x^2+1)^{3/2}|_3^8 = (2/3)(27-8) = 38/3$. Note that $9^{3/2} = (\sqrt{9})^3 = 3^3 = 27$ and that $4^{3/2} = 2^3 = 8$.

j) $\lim_{N\to\infty} \sum_{k=1}^N \frac{e^{k/N}}{N}$. This sum is of the form $\sum f(x_k)\Delta x$, where $f(x) = e^x$ and $x_k = k/N$. Taking the limit gives $\int_0^1 e^x dx$, which works out to $e^x|_0^1 = e - 1$.