M403K First Midterm Exam Solutions, February 5, 2002

1. An explosive shell is shot out of a cannon. Its height above the ground (in feet) is given by the function $f(x) = 160x - 16x^2$, where x is the time (in seconds) since the firing.

a) Find the velocity of the shell as a function of time.

$f'(x) = 160 - 32x$.

b) What is the shell's velocity at time $x = 4$? At that time, is it heading up or down?

$f'(4) = 160 - 4(32) = 32$. That is, the shell's height is *increasing* at 32 feet/second, which means it is still heading up.

c) The shell will hit the ground again at time $x = 10$. What will its velocity be at that time?

$f'(10) = 160 - 10(32) = -160$. This velocity is negative, meaning the shell is *falling* at 160 feet/second.

2. a) **From the following table**, estimate $f'(3)$. Indicate clearly how you obtain your answer:

x	$f(x)$
2.97	7.8791
2.98	7.9196
2.99	7.9599
3	8
3.01	8.0399
3.02	8.0796
3.03	8.1191

Compute $[f(3+h) - f(3)]/h$ for different values of h, and take the limit as $h \to 0$.

h	$f(3+h) - f(3)$	$[f(3+h) - f(3)]/h$
-0.03	-0.1209	4.03
-0.02	-0.0804	4.02
-0.01	-0.0401	4.01
0	0	undefined
0.01	0.0399	3.99
0.02	0.0796	3.98
0.03	0.1191	3.97

It's clear that the limit as $h \to 0$ is 4. So $f'(3) = 4$.

b) You receive a secret message saying that the function in part (a) was really $f(x) = 8\sqrt{x-2}$. **If this message is true**, what should $f'(3)$ equal? How does this compare to the answer you got in part (a)?

If the message is true, then $f'(x) = 4/\sqrt{x-2}$, so $f'(3) = 4$, in agreement with the answer to part (a). In fact, the function in part (a) really *is* $f(x) = 8\sqrt{x-2}$, tabulated to 4 decimal places.

3. Evaluate the following limits, if they exist (or write DNE if they do not).

a) $\lim_{x\to 2} \frac{x^2-3x+2}{x+2} = \frac{0}{4} = 0$

b) $\lim_{x\to 2} \frac{x^2-3x+2}{x-2} = \lim_{x\to 2} \frac{(x-2)(x-1)}{x-2} = \lim_{x\to 2}(x-1) = 1$

c) $\lim_{h\to 1} \frac{(1+h)^{14.5}-1}{h}$. This is the derivative of the function $f(x) = x^{14.5}$ at $x = 1$. Since $f'(x) = 14.5x^{13.5}$, $f'(1) = 14.5$.

d) $\lim_{x\to 2^-} \frac{x+3}{x-2}$. The numerator approaches 5, while the denominator approaches zero *from the negative side*, so the ratio goes to $-\infty$.

4. Take the derivatives of the following functions with respect to x. You do not need to simplify your answers:

a) $f(x) = (3x+1)^{13}$.

By the chain rule, $f'(x) = 13(3x+1)^{12} D_x(3x+1) = 39(3x+1)^{12}$.

b) $f(x) = (4+\sqrt{x})(x^2+7x+2)$. By the product rule, $f'(x) = (4+\sqrt{x})(2x+7) + (x^2+7x+2)[1/(2\sqrt{x})]$.

c) $f(x) = \frac{x-3}{2x+5}$

By the ratio rule, $f'(x) = \frac{(2x+5)-2(x-3)}{(2x+5)^2} = \frac{11}{(2x+5)^2}$.

d) $f(x) = (x+1)^5(x^2+(1/x))$

Combining the product rule and the chain rule, we get $f'(x) = 5(x+1)^4(x^2+(1/x)) + (x+1)^5(2x-1/x^2)$.

5. Nimbus Racing Brooms, Ltd. has just released their finest model, the Nimbus 2002. (Available at Quality Quiddich Supplies and fine stores everywhere.) Their marketing department has determined that the demand function is $x = 800 - 40p$, where x is the number of brooms sold (per month) and p is the price (in gold galleons). The cost function is $C(x) = 500 + 10x$.

a) Find the price $p(x)$ and the revenue $R(x)$ as a function of x.

Solving for p gives $p = (800 - x)/40 = 20 - \frac{x}{40}$, so $R(x) = xp = 20x - \frac{x^2}{40}$.

b) Compute the marginal cost, marginal revenue and marginal profit as a function of x.

$C'(x) = 10$, $R'(x) = 20 - \frac{x}{20}$, and $P'(x) = R'(x) - C'(x) = 10 - \frac{x}{20}$.

c) The factory is currently producing $x = 300$ brooms/month. To increase *revenue*, should the company increase or decrease production?

When $x = 300$, $R'(x) = 20 - \frac{300}{20} = 5$, so each additional broom will *increase* revenue by 5 galleons. To increase revenue, the company should increase production.

d) Likewise, if the company wants to increase *profit*, should it increase or decrease production?

When $x = 300$, $P'(x) = 10 - \frac{300}{20} = -5$, so each additional broom will *decrease* profits by 5 galleons. To increase profits, decrease production!