M403K Third Midterm Exam Solutions
April 11, 2002

1. Related rates.

Consider the curve $y^2 = x^3 + 1$, $y \geq 0$.

a) Find the slope of the line that is tangent to the curve at the point (2,3).

Take the derivative of the equation with respect to x: $2yy' = 3x^2$, so $y' = 3x^2/(2y) = 12/6 = 2$.

b) A particle is moving along the curve. Its x-coordinate is increasing at a rate of 10 units/second. How fast is y changing when $(x, y) = (2, 3)$?

There are two reasonably easy solutions. One is to use the result from (a): $dy/dt = (dy/dx)(dx/dt) = 2(10) = 20$ units/second.

The other method is to start from scratch, and take the derivative of the equation with respect to t:

$$2y\frac{dy}{dt} = 3x^2\frac{dx}{dt}.$$

Plugging in values of x, y and dx/dt gives $6(dy/dt) = 120$, so $dy/dt = 20$, as before.

Problem 2. L'Hopital's Rule Evaluate the following limits:

a) $\lim_{x\to\infty} \frac{15x^2 - 9}{x^3 + 3x^2 + 5} = \lim_{x\to\infty} \frac{30x}{3x^2 + 6x} = \lim_{x\to\infty} \frac{30}{6x + 6} = 0.$

b) $\lim_{x\to 3} \frac{x^2 - 9}{x - 3} = \lim_{x\to 3} \frac{2x}{1} = 6.$

c) $\lim_{x\to 2} \frac{\ln(x) - \ln(2)}{x - 2} = \lim_{x\to 2} \frac{1/x}{1} = \frac{1}{2}.$

d) $\lim_{x\to 0} \frac{e^x - 1}{x^2 + 1} = \frac{0}{1} = 0$. L'Hopital's rule does not apply here.

Problem 3. Elasticity of Demand

The demand x for a new toy depends on its price p via the demand equation

$$x = 1000e^{-p}.$$

a) Compute the elasticity of demand $E(p)$ as a function of p.

$$E(p) = \frac{p\frac{dx}{dp}}{x} = \frac{p(-1000e^{-p})}{1000e^{-p}} = -p.$$

b) For what values of p is the demand elastic? For what values of p is the demand inelastic?

When $p > 1$, $E < -1$ and the system is elastic. [Under these circumstances we should lower the price to increase revenue.]

When $p < 1$, $E > -1$ and the system is inelastic. [To raise revenue, raise the price].

c) What value of p will maximize revenue?

$p = 1$.

Problem 4. Horse sense

For the first two years of life, a pony's height $H(t)$ grows at a rate

$$H'(t) = 15 - 3t^2,$$

(where height is measured in inches and time in years). At age 1, the pony is 45 inches tall.

a) How tall was the pony at birth?

$H(t) = \int H'(t)dt = \int(15-3t^2)dt = 15t-t^3+C$. To evaluate the constant, use the fact that $H(1) = 45$, so $45 = 15 - 1 + C$, so $C = 31$. Now plug back in to get

$$H(t) = 15t - t^3 + 31.$$

So when t was zero, H was 31.

b) How tall will the pony be at age 2?

$H(2) = 15(2) - 2^3 + 31 = 53$ inches.

Problem 5. Indefinite integrals.

Evaluate the following integrals:

a) $\int (2x + e^x)dx = x^2 + e^x + C$

b) $\int xe^{x^2} dx = \frac{1}{2}e^{x^2} + C$. (Integrate by substitution with $u = x^2$.)

c) $\int \frac{\ln(x)}{x}dx = \frac{(\ln(x))^2}{2} + C$. (Integrate by substitution with $u = \ln(x)$.)

d) $\int (2x+1)^4 dx = \frac{(2x+1)^5}{10} + C$. (Integrate by substitution with $u = 2x + 1$.)

Problem 6. Area under a curve.

We are interested (OK, OK, your instructor is interested) in finding the area under the curve $y = 2x^2 + 1$ between $x = 1$ and $x = 4$.

a) Estimate this area using 3 rectangles. Your final answer should be an explicit number, like 13 or 152.

Each rectangle has width (4-1)/3 = 1. The three rectangles have height $f(2)$, $f(3)$ and $f(4)$, so the estimated total area is $f(2) + f(3) + f(4) = 9 + 19 + 33 = 61$. [If you used the function values at 1, 2 and 3 instead of 2, 3, and 4, I gave full credit. The answer then would be 31]

b) Estimate the area using N rectangles. You can leave your answer as a sum, like $\sum_{k=1}^{N} 4(\ln(N) - 3)/N$ (no, that's not the right answer). Everything in the sum needs to be clearly defined, but **YOU DO NOT NEED TO SIMPLIFY OR EVALUATE THE SUM.**

$\Delta x = (4-1)/N = 3/N$ and $a = 1$, so $x_k = a + k\Delta x = 1 + \frac{3k}{N}$. Thus $f(x_k) = 2(1 + \frac{3k}{N})^2 + 1$, and our estimated area, $\sum_{k=1}^{N} f(x_k)\Delta x$, works out to

$$\sum_{k=1}^{N} \left(2\left(1 + \frac{3k}{N}\right)^2 + 1 \right) \frac{3}{N}$$