

M328K First Midterm Exam, February 21, 2003

1. Using induction, prove the formula:

$$\sum_{k=1}^n k^3 = \frac{n^2(n+1)^2}{4}$$

2. As you know, the Fibonacci numbers  $f_n$  are defined by  $f_1 = 1$ ,  $f_2 = 1$  and, for  $n > 2$ ,  $f_n = f_{n-1} + f_{n-2}$ . Give a rigorous proof of the assertion: “ $f_n$  is divisible by 3 if and only if  $n$  is divisible by 4.” [Hint: Before writing down your proof, you may want to first determine which Fibonacci numbers are congruent to 1 (mod 3), which are congruent to 2 (mod 3), and which are divisible by 3. I’m sure you’ll see the patterns quickly enough.]

3. Greatest common factors:

- a) Find the greatest common factor of 66 and 52.  
b) Write this number explicitly as a linear combination of 66 and 52. For instance, if  $(66, 52)$  were equal to 24 (which it obviously isn’t!), you might write “ $24 = 3 \times 52 - 2 \times 66$ ”.  
c) What is the least common multiple of 66 and 52?

4. Congruences, Diophantine equations and the Chinese Remainder Theorem.

- a) Find all integer solutions to the equation  $25x + 38y = 1$ .  
b) Find a solution to the equation  $25x \equiv 1 \pmod{38}$ .  
c) Find a solution to the equation  $38x \equiv 1 \pmod{38}$ .  
d) Find a positive solution to the congruences  $x \equiv 5 \pmod{25}$ ,  $x \equiv 8 \pmod{38}$ .