M328K First Midterm Exam Solutions, February 21, 2003

1. Using induction, prove the formula:

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

We prove this by induction. The formula is true for the base case $n = 1$, since $\sum_{k=1}^{1} k^3 = 1^3 = 1 = 1^2(1+1)^2/4$. Now for the inductive step. We assume the formula is true for $n = m-1$ and show it works for $n = m$. If it is true for $n = m-1$, then

$$\sum_{k=1}^{m-1} k^3 = (m-1)^2 m^2/4 = (m^4 - 2m^3 + m^2)/4.$$

But then

$$\sum_{k=1}^{m} k^3 = m^3 + \sum_{k=1}^{m-1} k^3 = m^3 + (m^4 - 2m^3 + m^2)/4 = (m^4 + 2m^3 + m^2)/4 = m^2(m+1)^2/4,$$

which is the formula we needed to show.

2. As you know, the Fibonacci numbers f_n are defined by $f_1 = 1$, $f_2 = 1$ and, for $n > 2$, $f_n = f_{n-1} + f_{n-2}$. Give a rigorous proof of the assertion: " f_n is divisible by 3 if and only if n is divisible by 4." [Hint: Before writing down your proof, you may want to first determine which Fibonacci numbers are congruent to 1 (mod 3), which are congruent to 2 (mod 3), and which are divisible by 3. I'm sure you'll see the patters quickly enough.]

I claim that

$$\begin{cases} f_n \equiv 0 \pmod 3 & \text{if } n \equiv 0 \text{ or } 4 \pmod 8 \\ f_n \equiv 1 \pmod 3 & \text{if } n \equiv 1, 2, \text{ or } 7 \pmod 8 \\ f_n \equiv 2 \pmod 3 & \text{if } n \equiv 3, 5, \text{ or } 6 \pmod 8. \end{cases}$$

If we can prove this, then the original assertion follows, since being divisible by 4 is the same as being congruent to 0 or 4 modulo 8.

We prove the claim by (generalized) induction. It is true for the base cases $n = 1$ and $n = 2$, since $f_1 = f_2 = 1$. Now, for $m > 2$, suppose that it is true for all values of n up to $m - 1$. We must show it is true for $n = m$. There are 8 cases to check:

If $m \equiv 0 \pmod 8$, then $m - 2 \equiv 6 \pmod 8$ and $m - 1 \equiv 7 \pmod 8$, so $f_{m-2} \equiv 2 \pmod 3$ and $f_{m-1} \equiv 1 \pmod 3$, so $f_m = f_{m-2} + f_{m-1} \equiv 1 + 2 \equiv 0 \pmod 3$.

If $m \equiv 1 \pmod 8$, then $m - 2 \equiv 7 \pmod 8$ and $m - 1 \equiv 0 \pmod 8$, so $f_{m-2} \equiv 1 \pmod 3$ and $f_{m-1} \equiv 0 \pmod 3$, so $f_m = f_{m-2} + f_{m-1} \equiv 1 + 0 = 1 \pmod 3$.

Similarly, if $m \equiv 2 \pmod 8$, then $f_{m-2} \equiv 0 \pmod 3$ and $f_{m-1} \equiv 1 \pmod 3$, so $f_m = f_{m-2} + f_{m-1} \equiv 0 + 1 \equiv 0 \pmod 3$.

If $m \equiv 3 \pmod 8$, then $f_{m-2} \equiv 1 \pmod 3$ and $f_{m-1} \equiv 1 \pmod 3$, so $f_m \equiv 2 \pmod 3$.

If $m \equiv 4 \pmod 8$, then $f_{m-2} \equiv 1 \pmod 3$ and $f_{m-1} \equiv 2 \pmod 3$, so $f_m \equiv 0 \pmod 3$.

If $m \equiv 5 \pmod 8$, then $f_{m-2} \equiv 2 \pmod 3$ and $f_{m-1} \equiv 0 \pmod 3$, so $f_m \equiv 2 \pmod 3$.

If $m \equiv 6 \pmod 8$, then $f_{m-2} \equiv 0 \pmod 3$ and $f_{m-1} \equiv 2 \pmod 3$, so $f_m \equiv 2 \pmod 3$.

If $m \equiv 7 \pmod 8$, then $f_{m-2} \equiv 2 \pmod 3$ and $f_{m-1} \equiv 2 \pmod 3$, so $f_m \equiv 4 \equiv 1 \pmod 3$, and then we're done.

3. Greatest common factors:

a) Find the greatest common factor of 66 and 52.

By the Euclidean algorithm: 66 = 1(52) + 14; 52 = 3(14) + 10; 14 = 1(10)+4; 10 = 2(4) + 2; 4 = 2(2). So (66,52)=2.

b) Write this number explicitly as a linear combination of 66 and 52. For instance, if (66,52) were equal to 24 (which it obviously isn't!), you might write "$24 = 3 \times 52 - 2 \times 66$".

Working through the algorithm, we have 14 = 66 - 52; 10 = 52 - 3(14) = 4(52) - 3(66); 4 = 14 - 10 = 4(66) - 5(52); and finally our answer: 2 = 10 - 2(4) = 14(52) - 11(66).

c) What is the least common multiple of 66 and 52?

$[66, 52] = 66 \times 52/(66, 52) = 66 \times 52/2 = 1716.$

4. Congruences, Diophantine equations and the Chinese Remainder Theorem.

a) Find all integer solutions to the equation $25x + 38y = 1$.

By inspection, or by the Euclidean algorithm, you can see that $-3(25) + 2(38) = -75 + 76 = 1$. Thus ALL integer solutions are of the form

$$x = -3 + 38n, \qquad y = 2 - 25n,$$

where n is an arbitrary integer.

b) Find a solution to the equation $25x \equiv 1 \pmod{38}$.

This is the same question, and the answer is $x = -3$ (or $x = 35$, or any number that is congruent to -3 modulo 38).

c) Find a solution to the equation $38x \equiv 1 \pmod{38}$.

$y = 2$ (or anything congruent to 2 modulo 25).

d) Find a positive solution to the congruences $x \equiv 5 \pmod{25}$, $x \equiv 8 \pmod{38}$.

Since $76 \equiv 1 \pmod{25}$ and $76 \equiv 0 \pmod{38}$, and since $-75 \equiv 0 \pmod{25}$ and $-75 \equiv 1 \pmod{38}$, $x = 5(76) + 8(-75) = -220$ is congruent to 5 modulo 25 and 8 modulo 38. However, we wanted a POSITIVE solution, so we add an arbitrary multiple of (25)(38)=950. The smallest solution is $-220 + 950 = 730$.