M340L Third Midterm Exam, April 7, 2003

1. The following two 4×4 matrices are row-equivalent:

$$A = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 1 & 3 & 8 & 0 \\ 2 & 4 & 10 & 1 \\ 3 & 5 & 12 & 0 \end{pmatrix}; \qquad B = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

a) What is the rank of A?

b) Find a basis for the column space of A.

c) Find a basis for the null space of A.

d) Find a basis for the row space of A.

2. Let $V = \text{Span}\{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 5 \\ 8 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}\} \subset \mathbf{R}^4$.

a) What is the dimension of V?

b) Find a basis for V.

c) Let W be the subspace of $P_3[t]$ spanned by the polynomials $1+t+t^2+t^3$, $1+2t+3t^2+4t^3$, $1+3t+5t^2+8t^3$ and $4+3t+2t^2+t^3$. What is the dimension of W? Find a basis for W.

3. Consider the following basis for $\mathbf{R}^3$:

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \\ 7 \end{pmatrix}, \begin{pmatrix} 6 \\ 12 \\ 13 \end{pmatrix} \right\}$$

a) Find the change-of-basis matrix P_{EB} that converts from coordinates in the B basis to coordinates in the standard (E) basis.

b) Fina the change-of-basis matrix P_{BE} that converts from coordinates in the standard basis to coordinates in the B basis.

c) Let $x = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$. Compute $[x]_B$.

d) Consider the basis $\mathcal{C} = \{1+2t+2t^2,\ 3+7t+7t^2,\ 6+12t+13t^2\}$ for the vector space $P_2[t]$. Find the coordinates of the polynomial $3+t+4t^2$ in this basis.

4. a) Are the polynomials $1+t+t^2$, $1+3t+5t^2$ and $7+4t+t^2$ linearly independent?

b) Do the polynomials $1 + t^2$, $2t + 5t^2$ and $t + 2t^2$ span $P_2[t]$?

5. Indicate whether each of these statements is true or false. If a statement is sometimes true and sometimes false, write "false". You do NOT have to justify your answers. There is no penalty for wrong answers, so go ahead and guess if you are unsure of your answer.

a) The rank of a matrix is the dimension of its row space.

b) Let v_1, v_2, v_3 be vectors in a vector space V. For $\text{Span}\{v_1, v_2, v_3\}$ to be a subspace of V, the vectors must be linearly independent.

c) If A and B are row-equivalent matrices, then $Col(A) = Col(B)$.

d) If A and B are row-equivalent matrices, then $Null(A) = Null(B)$.

e) If A and B are row-equivalent matrices, then $Row(A) = Row(B)$.

f) Let $\{v_1, v_2, v_3, v_4\}$ be a collection of four vectors in P_2. One of the v_i's can be written as a linear combination of the others.

g) If a collection of five polynomials spans P_3, then it forms a basis for P_3.

h) Every change-of-basis matrix is invertible.

i) If a linearly dependent set of vectors spans $\mathbf{R}^3$, then there must be at least 4 vectors in the set.

j) If $\{v_1, \ldots, v_k\}$ is a linearly independent set of vectors in V, and $\{w_1, \ldots, w_n\}$ is a spanning set for V, then $n > k$.