M340L Practice Final Exam
Originally given May 13, 1995

**Problem 1:** Find all solutions (if any) to the system of equations. Express your answer in vector parametric form.

$$\begin{aligned} x_1 \qquad + 2x_3 + 3x_4 &= 6 \\ 2x_1 + 2x_2 + x_3 - 3x_4 &= 2 \\ 4x_1 + 2x_2 + 5x_3 + 3x_4 &= 14 \end{aligned}$$

**Problem 2:** $T$ is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^4$ defined by $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_1 - x_2 \\ x_2 \\ x_1 + x_2 \end{pmatrix}$.

a) Find the matrix of this linear transformation.

b) Is $T$ 1-1? If not, find a nonzero vector $\mathbf{x}$ such that $T(\mathbf{x}) = 0$.

c) Is $T$ onto? If not, find a nonzero vector $\mathbf{y}$ such that $\mathbf{y}$ is not in the range of $T$.

**Problem 3:** a) Compute the determinant of the matrix

$$A = \begin{pmatrix} 4 & 8 & 8 & 8 & 5 \\ 0 & 1 & 0 & 0 & 0 \\ 6 & 8 & 8 & 8 & 7 \\ 0 & 8 & 8 & 3 & 0 \\ 0 & 8 & 2 & 0 & 0 \end{pmatrix}$$

b) Is $A$ invertible? Why or why not?

c) What is the rank of $A$?

**Problem 4:** Let $\mathcal{E} = \{1, t, t^2\}$ be the standard basis for $\mathbb{P}_2$.
Let $\mathcal{B} = \{1 + t + t^2, 1 + 2t + 3t^2, 1 + 4t + 9t^2\}$ be another basis.
Let $T : \mathbb{P}_2 \to \mathbb{P}_2$ be the linear transformation $T(p(t)) = p(t) + t(dp(t)/dt)$.

a) Find the matrix of $T$ relative to the standard basis $\mathcal{E}$. Call this matrix $A$.

b) Find the matrix of $T$ relative to the basis $\mathcal{B}$. Call this matrix $B$.

c) Write down the change-of-basis matrix from $\mathcal{B}$ to $\mathcal{E}$. Call this matrix $P$.

d) Write an equation expressing $B$ in terms of $A$ and $P$.

**Problem 5:** The following matrices are row-equivalent:

$$A = \begin{pmatrix} 1 & 1 & 5 & 1 & 8 \\ 1 & 2 & 7 & 0 & 7 \\ 1 & 3 & 9 & 0 & 10 \\ 2 & 4 & 14 & 1 & 18 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

a) Find a basis for the row space of $A$.

b) Find a basis for the column space of $A$.

c) Find a basis for the null space of $A$.

**Problem 6:** Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

a) Find all the real eigenvalues of $A$ and the corresponding eigenvectors.

b) Find all *complex* eigenvalues and corresponding eigenvectors of $A$.

c) Find an invertible matrix $P$ and a diagonal matrix $D$ such that $A = PDP^{-1}$. [Warning: $P$ and $D$ may not be real].

**Problem 7:** In contrast to predator-prey relationships, it can sometimes happen that two different kinds of animals actively help each other. This is called symbiosis. Since I can't think of any realistic examples, I'll illustrate with mythical animals.

Griffins and Dragons live in the Enchanted Forest. Dragons can live without griffins, but griffins cannot live without dragons. The number of dragons $D$ and griffins $G$ in the forest each year is determined by the populations the previous year, according to the formulas:

$$\begin{aligned} D_{k+1} &= 1.5D_k + G_k, \\ G_{k+1} &= D_k \end{aligned}$$

The eigenvalues of the matrix $\begin{pmatrix} 1.5 & 1 \\ 1 & 0 \end{pmatrix}$ are 2 and $-.5$, with corresponding eigenvalues $(2, 1)$ and $(1, -2)$.

a) If in year 0 there are 25 dragons and no griffins, what will the populations be in year $k$? (Don't worry about your answers being fractional. Mythical animals don't have to come in whole units).

b) In the long run, will the populations grow, shrink, or approach a nonzero equilibrium value?

c) After a long time, approximately what will the ratio of dragons to griffins be?

**Problem 8:** Let $W$ be the subspace of $\mathbb{R}^4$ spanned by the vectors $(1,1,1,1)^T$, $(1,2,2,3)^T$, and $(5,7,7,9)^T$.

a) What is the dimension of $W$?

b) Find an orthonormal basis for $W$.

**Problem 9:** Let $W$ be the subspace of $\mathbb{R}^4$ spanned by $u_1 = (1,0,1,0)^T$ and $u_2 = (4,3,-4,3)^T$. (Note that $u_1$ and $u_2$ are orthogonal).

a) Find the orthogonal projection of the vector $(3,0,3,5)^T$ onto the plane $W$.

b) Find the distance from $W$ to the point $(3,0,3,5)$.

b) Find a least-squares solution to the system of equations

$$\begin{pmatrix} 1 & 4 \\ 0 & 3 \\ 1 & -4 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 3 \\ 5 \end{pmatrix}$$

**Problem 10. True of False**

a) A $6 \times 5$ matrix cannot have a pivot position is every row.

b) Let $v_1, v_2$ and $v_3$ be vectors in $\mathbb{R}^3$. If none of these vectors is a multiple of one of the others, then the set is linearly independent.

c) If a system $Ax = b$ has more than one solution, then $Ax = 0$ has more than one solution.

d) If $AB = BA$ and $A$ is invertible, then $A^{-1}B = BA^{-1}$.

e) If $AB = 0$, then either $A = 0$ or $B = 0$.

f) For every real matrix $A$, $\det(A^T A) \geq 0$.

g) Let $S$ be a set of vectors in a vector space $V$. If $\mathrm{Span}(S) = V$, then a subset of $S$ is a basis for $V$.

h) A change-of-basis matrix is always invertible.

i) If a square matrix $A$ is diagonalizable, then its columns are linearly independent.

j) If $W$ is a subspace of $\mathbb{R}^n$, then $W$ and $W^{\perp}$ have no nonzero vectors in common.