M340L Midterm Exam (from November 4, 1993)

Problem 1: Consider the linear operator $L : P_2 \to P_2$, $L(p(t)) = (t+1)p'(t) - 2p(t)$. As usual, P_2 is the space of 2nd order polynomials in the variable t.

a) What is the matrix of L relative to the basis $S = \{t^2, t, 1\}$?

b) Find the dimension of the kernel of L and the dimension of the range of L.

c) Find a basis for the kernel of L. Also, find a basis for the range of L.

Problem 2 This problem concerns changing bases in $\mathbf{R}^3$. Let S be the standard basis. Let $T = Y_1, Y_2, Y_3$ be another basis, where $Y_1 = (1,2,3)$, $Y_2 = (1,1,1)$, $Y_3 = (1,3,4)$.

a) Find a matrix P that converts from the T basis to the S basis. That is, so that for any vector X, $[X]_S = P[X]_T$.

b) Find a matrix Q that converts from the S basis to the T basis. That is, so that for any vector X, $[X]_T = Q[X]_S$.

c) There is a linear operator L whose matrix, relative to the S basis, is $A = \begin{pmatrix} 0 & 1 & 0 \\ -4 & 7 & -2 \\ -5 & 7 & 1 \end{pmatrix}$. Find the matrix of L relative to the T basis.

Problem 3. Let $A = \begin{pmatrix} 1 & 3 & 2 & 8 & 4 \\ 2 & 6 & 5 & 17 & 9 \\ 3 & 9 & 0 & 6 & 0 \end{pmatrix}$

a) What is the rank of A?

b) Find a basis for the null space of A.

c) Find a basis for the row space of A.

Problem 4. Let V be the span of X_1, X_2 and X_3, where $X_1 = (1,0,1,0)$, $X_2 = (1,1,1,1)$ and $X_3 = (1,2,3,4)$.

a) Find an orthogonal basis for V.

b) Find an orthonormal basis for V.

Problem 5. True of False

a) The map $L : \mathbf{R}^2 \to \mathbf{R}^2$, $L(x, y) = (x+y, y-1)$, is a linear transformation.

b) The map $L : \mathbf{R}^2 \to \mathbf{R}^2$, $L(x, y) = (x + y^2, y)$, is a linear transformation.

c) The map $L : \mathbf{R}^2 \to \mathbf{R}^2$, $L(x, y) = (3x+y, y-5x)$, is a linear transformation.

d) Let L be a linear transformation from $\mathbf{R}^4$ to $\mathbf{R}^5$. If the rank of L is 3, then the kernel of L is 1-dimensional.

e) Let L be a linear transformation from $\mathbf{R}^5$ to $\mathbf{R}^3$. If the kernel of L is 2 dimensional, then L is onto.

f) Let L be a linear transformation from $\mathbf{R}^4$ to $\mathbf{R}^5$. If L is 1-1, then L is onto.

g) If a set of vectors is orthonormal, then it is a basis.

h) If a set of vectors is orthonormal, then it is linearly independent.

i) If a matrix is wider than it is tall (e.g. a 3×5 matrix), then the row rank is greater than the column rank.

j) Let L be a linear transformation from P_3 to itself. The solutions (in P_3) to the equation $L(p) = 0$ form a vector space.