M362K Final Exam Solutions. May 13, 2004

Problem 1. Delusional politicians

1% of the population suffers from "delusions of grandeur", a person's belief that he is far, far more important than the facts would indicate. 5% of the people with such delusions run for public office, but only 0.1% of people without delusions of grandeur run for public office. [Note: these numbers are made up, but the psychosis is real. I'll leave it to you to decide whether the diagnosis applies to anybody you know.]

a) What is the probability of a randomly chosen person running for office?

Let D be the event "has delusions of grandeur" and let R be the event "runs for office". We're told that $P(D) = 0.01$, $P(R|D) = 0.05$, $P(R|D^c) = 0.001$. Then $P(R) = P(R \cap D) + P(R \cap D^c) = (0.01)(0.05) + (0.99)(0.001) = 0.0005 + 0.00099 = 0.00149 = 0.149\%$.

b) Given that a person is running for office, what is the probability of his having delusions of grandeur?

$$\frac{P(R \cap D)}{P(R)} = \frac{0.0005}{0.00149} = \frac{50}{149} \approx 0.33557 = 33.557\%$$

Problem 2. 5 Crowns

5 Crowns is a real card game that my children enjoy. There are 5 suits (clubs, diamonds, hearts, spades and stars) and the cards range in value from 3 to King (no aces or 2s). Suppose a player is dealt 8 cards.

a) What is the probability that he has no kings? Give an exact answer, which you can leave in the form of factorials or binomial coefficients.

There are 55 cards, of which 5 are kings. There are $\binom{50}{8}$ ways to choose a hand with no kings, and $\binom{55}{8}$ ways to choose a hand, so the probability of having no kings is $\binom{50}{8}/\binom{55}{8} \approx 0.4409$.

b) What is the probability that he has 3 clubs, 2 diamonds, one heart, one spade and one star?

There are $\binom{11}{3}$ ways to choose the clubs, $\binom{11}{2}$ ways to choose the diamonds, 11 ways to choose the heart, 11 ways to choose the spade, and 11 ways to choose the star, hence $\binom{11}{3}\binom{11}{2}11^3$ ways to choose the hand, for a probability of $\binom{11}{3}\binom{11}{2}11^3/\binom{55}{8}$.

c) What is the probability that he has 4 pair?

There are $\binom{11}{4}$ ways to pick which 4 ranks will have pairs, and $\binom{5}{2}$ ways to choose the suits for each rank, so the answer is

$$\frac{\binom{11}{4}\binom{5}{2}^4}{\binom{55}{8}}$$

Extra credit (write answer on back): Approximate the answer to (a) using the binomial or Poisson distribution. Your final answer should be numerical.

(Binomial) Each card has a 1/11 chance of being a king. If we did sampling WITH replacement, the probability of drawing 8 non-kings would be $(10/11)^8 \approx 0.4665$. Note that this is a little too high, since sampling without replacement depletes the deck of non-kings and increases the chance of getting a king in the first 8 cards.

(Poisson) On the average you expect $8/11 = 0.72727$ kings, so the probability of getting none is around $e^{-0.727} \approx 0.4832$.

Problem 3. Joint distributions

X and Y are discrete random variables whose joint pdf is given in the table :

$Y\backslash X$	1	2	5	f_Y
1	.12	.03	.15	.30
2	.20	.05	.25	.50
3	.08	.02	.10	.20
f_X	.40	.10	.50	

a) Find the marginal probability density functions $f_X(x)$ and $f_Y(y)$ for all possible values of X and Y, respectively.

[See above. The table was expanded to include the marginals]

b) Find the conditional probability $P(Y = 3|X = 2)$. Are the events "$Y = 3$" and "$X = 2$" independent? Are X and Y independent random variables?

$P(Y = 3|X = 2) = f_{X,Y}(2,3)/f_X(2) = .02/.10 = .20$. Since this equals $f_Y(3)$, these events are independent. A quick check shows that $f_{X,Y}(x,y) = f_X(x)f_Y(y)$ in all cases, so X and Y are independent random variables.

c) Let $Z = X + Y$. Compute the pdf $f_Z(z)$.

$f_Z(2) = f_{X,Y}(1,1) = 0.12$, $f_Z(3) = f_{X,Y}(1,2) + f_{X,Y}(2,1) = 0.23$, $f_Z(4) = f_{X,Y}(1,3) + f_{X,Y}(2,2) = 0.13$, $f_Z(5) = f_{X,Y}(2,3) = 0.02$, $f_Z(6) = f_{X,Y}(5,1) = 0.15$, $f_Z(7) = f_{X,Y}(5,2) = 0.25$, $f_Z(8) = f_{X,Y}(5,3) = 0.10$, and all other values are impossible.

Problem 4. Light bulbs

The lifetime of a 60 watt incandescent light bulb is given by the exponential distribution, with mean 1000 hours.

a) Find the probability that a randomly chosen light buld last less than 50 hours.

$F_X(50) = 1 - e^{-50/1000} = 1 - e^{-0.05} \approx 0.04877$. Note that the continuity correction does NOT apply, since we are already dealing with a continuous random variable.

b) A Christmas light display consists of 1000 light bulbs, whose lifetimes are assumed to be independent. Let X be the number of light bulbs that burn out in the first 50 hours. Which distribution describes X? Give an exact formula for $f_X(x)$.

This is binomial, with $n = 1000$ and $p = 1 - e^{-0.05} \approx 0.04877$. $f_X(x) = \binom{1000}{x} p^x (1-p)^{1000-x}$.

c) Approximate the probability that $X \leq 45$ using the normal distribution.

Since we want all integers up to and including 45, we set the cutoff at 45.5. Now $\mu = np = 48.77$, while $\sigma = \sqrt{np(1-p)} = 6.811$. Our Z score is then $(45.5 - 48.77)/6.811 \approx -0.48$, and the probability is $F_Z(z) = 0.3156$.

Problem 5. Misbehaving kids

Like all children, my kids sometimes misbehave. Being children of a mathematician, they misbehave according to independent Poisson processes. Allan misbehaves an average of 3 times per hour, Rina an average of 2 times per hour, and Jonathan an average of 1.5 times per hour.

a) In one hour, what is the probability that Jonathan doesn't misbehave at all? What is the probability that Allan misbehaves 2 or fewer times.

Jonathan's misbehaviors are Poisson with mean 1.5, so $P(J = 0) = e^{-1.5} \approx 0.223$. Allan's are Poisson with mean 3, so $P(A \leq 2) = P(A = 0) + P(A = 1) + P(A = 2) = (3^0 + 3^1 + 3^2/2)e^{-3} \approx 0.423$.

b) In a 15 minute period, what is the probability that there will be exactly one instance of misbehavior among the kids?

In 15 minutes, the average number of misbehaviors is $\lambda = (3{+}2{+}1.5)/4 = 1.625$, so P(one misbehavior) $= 1.625e^{-1.625} \approx 0.32$.

c) Let X be the total number of misbehaviors in 2 hours (for all three kids put together). Find the mean and standard deviation of X.

This is Poisson with $\lambda = 2(3 + 2 + 1.5) = 13$. The mean is $\lambda = 13$ and the standard deviation is $\sqrt{\lambda} = \sqrt{13} \approx 3.6$.

Problem 6. Manipulating random variables (10 points)

Let X be a continuous random variable, uniformly distributed between 1 and 2. Let $Y = X^2$.

a) Find $f_Y(y)$ for all values of y.

Since X is never negative,

$$f_Y(y) = f_X(\sqrt{y})/2\sqrt{y} = \begin{cases} 1/2\sqrt{y} & \text{if } 1 < y < 4 \\ 0 & \text{otherwise} \end{cases}$$

b) Find $E(Y)$ and $Var(Y)$. [Note: it is possible to do (b) without first doing (a).]

You could integrate $yf_Y(y)$ and $y^2 f_Y(y)$ to get $E(Y)$ and $E(Y^2)$, but it's easier to compute

$$E(Y) = E(X^2) = \int_1^2 x^2 dx = 7/3$$
$$E(Y^2) = E(X^4) = \int_1^2 x^4 dx = 31/5$$

$Var(Y) = E(Y^2) - (E(Y))^2 = 34/45$.

Problem 7. Pennies

A penny is approximately 1 gram. That is, the weight of a (random) penny is a continuous random variable with mean 1.00 grams and standard deviation 0.04 grams. A roll of pennies contains 50 pennies, whose weights are independent. Let X be the weight of a roll of pennies, in grams.

a) Find $E(X)$ and $Var(X)$.

$E(X) = 50$ grams, and $Var(X) = 50(0.04)^2 = 0.08$, so $\sigma = \sqrt{0.08} \approx 0.2828$ grams.

b) Using the normal distribution, estimate the probability that $49.5 < X < 50.5$.

The Z score of 50.5 is $(50.5 - 50)/0.2828 \approx 1.77$ and that of 49.5 is -1.77, so $P(49.5 < X < 50.5) \approx F_Z(1.77) - F_Z(-1.77) = 0.9616 - 0.0384 = 0.9232$.