M362K First Midterm Exam Solutions. February 13, 2004

Problem 1. Sample spaces and events

In the game of "third chance" (which I just made up), a player rolls an ordinary 6-sided die. If the die comes up 1 or 2, he rolls a second die. If the second die comes up 1, he rolls a third die. His score is the sum of all the dice rolled. For instance, he might score a 5 by rolling a 5 on his first throw, by rolling a 1 and then a 4, by rolling a 2, then a 1, and then a 2, or by several other means. [To fix notation once and for all, let's describe those three scenarios as (5), (1,4) and (2,1,2)]. Let A be the event "player rolls exactly 2 dice" and let B be the event "player scores 6".

a) List all the points in the sample space. How many are there?

$S = \{(1,1,1),(1,1,2),(1,1,3),(1,1,4),(1,1,5),(1,1,6),(1,2),(1,3),$ $(1,4),(1,5),(1,6),(2,1,1),(2,1,2),(2,1,3),(2,1,4),(2,1,5),(2,1,6),$ $(2,2),(2,3),(2,4),(2,5),(2,6),(3),(4),(5),(6)\}$. 26 points in all.

b) List the points in the event A. Separately, list the points in the event B.

$$A = \{(1,2),(1,3),(1,4),(1,5),(1,6),(2,2),(2,3),(2,4),(2,5),(2,6)\}$$

$$B = \{(1,1,4),(1,5),(2,1,3),(2,4),(6)\}$$

There are 10 points in A and 5 points in B.

c) List the points in the event $A \cap B$. List the points in the event $A \cup B$.

$A \cap B = \{(1,5),(2,4)\}$,
$A \cup B = \{(1,1,4),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1,3),$
$(2,2),(2,3),(2,4),(2,5),(2,6),(6)\}$

Problem 2. Continuous distributions

A random number is chosen from the interval $0 \le x \le 1$ with probability density function $f(x) = 2x$. Let A be the event "number is between 1/3 and 2/3", and let B be the event "number is greater than 1/2".

a) Find the probability of A: $P(A) = \int_{1/3}^{2/3} 2x dx = 1/3$.

b) Find the probability of B: $P(B) = \int_{1/2}^{1} 2x dx = 3/4$.

c) Find the conditional probability $P(A|B)$.

$P(A \cap B) = \int_{1/2}^{2/3} 2x dx = 7/36$, so $P(A|B) = P(A \cap B)/P(B) = (7/36)/(3/4) = 7/27$.

Problem 3. Independence

A gambler rolls two fair dice (one red and one green). Let A be the event "the total roll is a 6 or an 8", let B be the event "The red die shows a 1, 2 or 3", and let C be the event "The green die shows a 1, 2, or 3".

a) Compute the probabilities of A, B, C, $A \cap B$, $A \cap C$, $B \cap C$ and $A \cap B \cap C$.

Our sample space has 36 points, all equally likely, so finding probabilities pretty much boils down to counting points in events. $P(A) = 10/36 = 5/18$, $P(B) = 3/6 = 1/2$, $P(C) = 3/6 = 1/2$, $P(B \cap C) = 1/4$. Note that $A \cap B = \{(1,5),(2,4),(2,6),(3,3),(3,5)\}$ so $P(A \cap B) = 5/36$. A similar count gives $P(A \cap C) = 5/36$. Since $A \cap B \cap C = \{(3,3)\}$, $P(A \cap B \cap C) = 1/36$.

b) Are A and B independent events? Are A and C independent events? Are B and C independent events?

Since (by part (a)) $P(A \cap B) = P(A)P(B)$, A and B are independent. Since $P(A \cap C) = P(A)P(C)$, A and C are independent. Since $P(B \cap C) = P(B)P(C)$, B and C are independent.

c) Are A, B and C independent events?

No. $P(A \cap B \cap C) = 1/36 \neq P(A)P(B)P(C) = 5/72$

Problem 4. Stolen elections

In the race for Sivart County Commissioner, the candidate of the Loot and Pillage (L&P) Party has hired some hackers to break into the election computers and fix the result. The hackers have a 20% chance of breaking into the system. If they get in, they have a 90% chance of stealing enough votes to win the election. If they don't get in, the L&P candidate has a 15% chance of winning anyway.

a) What is the probability that the L&P candidate will win the election.

Let A be the event of a successful break in, and let B be the event that the L&P candidate wins the election.

$P(B) = P(B|A)P(A) + P(B|A^c)P(A^c) = (.9)(.2) + (.15)(.8) = .18 + .12 = .3$, or 30%.

b) Suppose that the L&P candidate wins. What is the probability that the hackers succeeded in breaking into the system?

$P(A|B) = P(A \cap B)/P(B) = .18/.3 = .6$, or 60%.