M362K Second Midterm Exam. March 12, 2004

Problem 1. Bridge

A bridge hand consists of 13 cards from a standard 52 card deck.

a) How many bridge hands contain exactly five spades?

There are $\binom{13}{5}$ ways to pick the spades and $\binom{39}{8}$ ways to pick the other cards, hence $\binom{13}{5}\binom{39}{8}$ possible hands.

b) How many bridge hands contain all 4 aces?

There are $\binom{4}{4} = 1$ ways to pick the aces and $\binom{48}{9}$ ways to pick the other cards, hence $\binom{48}{9}$ possible hands.

c) How many bridge hands contain exactly 5 spades and all 4 aces?

There are $\binom{4}{4} = 1$ ways to pick the aces, $\binom{12}{4}$ ways to pick the other spades, and $\binom{36}{5}$ ways to pick the other cards, or $\binom{12}{4}\binom{36}{5}$ possible hands.

d) How many bridge hands contain either 4 aces or 5 spades (or both)?

#(5 spades or 4 aces) = #(5 spades) + #(4 aces) - #(4 aces and 5 spades), or $\binom{13}{5}\binom{39}{8} + \binom{48}{9} - \binom{12}{4}\binom{36}{5}$.

Problem 2. Crooked dice For this problem, please evaluate all answers numerically. A gambler has two dice in his pocket. One is fair, and the other is weighted to give a "6" 50% of the time.

a) If he rolls the fair die 5 times, what is the probability of his getting a "6" exactly twice? If he rolls the weighted die 5 times, what is the probability of getting a "6" exactly twice.

The first situation is binomial with $p = 1/6$, and the answer is $\binom{5}{2}(1/6)^2(5/6)^3 \approx .1608$

The second situation is binomial with $p = 1/2$ and the answer is $\binom{5}{2}(1/2)^2(1/2)^3 = .3125$.

b) The gambler randomly selects a die (50-50 chance of each die being chosen) and rolls it 5 times. What is the probability that he gets a "6" exactly twice.

Let A be the event "chooses weighted die" and let B be the event "rolls two 6's". $P(B) = P(A)P(B|A)+P(A^c)P(B|A^c) = (1/2)0.3125+(1/2)0.1608 \approx 0.2367$.

c) If the gambler picks a die at random, rolls it 5 times, and gets a "6" twice, what is the probability that he was rolling the weighted die?

By Bayes' theorem, $P(A|B) = P(A)P(B|A)/P(B) = (1/2)0.3125/0.2367 \approx 0.6603$.

Problem 3. Urns

Two identical urns are filled with balls. One urn has 30 black balls and 20 white balls. The other urn has 2 black balls and 8 white balls.

a) An urn is picked at random, and then a ball is drawn (at random) from that urn. What is the probability that the ball is black?

If one urn is picked, the chances are 30/50=.6, while if the other urn is picked, the chances are 2/10=.2. Since each urn has a 50% chance of being picked, the chance of getting a black ball is $(1/2).6 + (1/2).2 = .4$

b) An urn is picked at random, and then four balls are drawn from that urn, without replacement. What is the probability that exactly two of the four balls are black?

Whichever urn is picked, the problem is hypergeometric. In one case, there is a $\binom{30}{2}\binom{20}{2}/\binom{50}{4}$ chance of getting 2 black and 2 white, while in the other case there is a $\binom{2}{2}\binom{8}{2}/\binom{10}{4}$ chance. Since each urn has a 50% chance of being chosen, the overall chance of getting exactly 2 black balls is

$$\frac{1}{2}\left[\frac{\binom{30}{2}\binom{20}{2}}{\binom{50}{4}} + \frac{\binom{2}{2}\binom{8}{2}}{\binom{10}{4}}\right]$$

c) An urn is picked at random, and then four balls are drawn from that urn, WITH replacement. (That is, a ball it drawn and thrown back in, then another ball is drawn and thrown back in, etc.) What is the probability that exactly two of the four balls are black?

This is similar, only with binomial instead of hypergeometric:

$$\frac{1}{2}\left[\binom{4}{2}(3/5)^2(2/5)^2 + \binom{4}{2}(1/5)^2(4/5)^2\right]$$

Problem 4. Political grab bag

a) How many words (nonsense or real) can by made by rearranging the letters of the word GERRYMANDER? How many begin and end with the letter R? How many begin with the letters DEM?

This problem is multinomial. There are 3 Rs, 2 Es, and six single letters in the word GERRYMANDER, so there are 11!/(3!2!1!1!1!1!1!1!) rearrangements. Simplifying, this is 11!/(3!2!), or 3,326,400.

If we put an R at the beginning and an R at the end, we are left with scrambling the other 9 letters, of which there are 2 Es and 7 single letters. There are $9!/2! = 181,440$ ways to do this.

If we start with DEM, we have to scramble the remaining 8 letters, of which there are 3 Rs and 5 single letters. There are $8!/3! = 6,720$ ways of doing this.

I'll let you decide whether there is any real-world significance to these numbers!

b) A government official is supposed to award 5 defense contracts. There are three companies competing for these contracts. If the contracts are all different, in how many ways can the contracts be awarded? If the contracts are all the same (so all a company cares about is how MANY contracts it gets), how many ways are there of awarding contracts?

If the contracts are different, then there are 3 ways to assign each contract, and $3^5 = 243$ possiblities in all.

If the contracts are all the same, then we have $\binom{5+3-1}{5} = \binom{7}{5} = 21$ possibilities. If (a, b, c) denotes company A getting a contracts, company B getting b contracts, and company C getting c contracts, then the 21 possibilities are: (0,0,5), (0,1,4), (0,2,3), (0,3,2) (0,4,1), (0,5,0), (1,0,4), (1,1,3), (1,2,2), (1,3,1), (1,4,0), (2,0,3), (2,1,2), (2,2,1), (2,3,0), (3,0,2), (3,1,1), (3,2,0), (4,0,1), (4,1,0), (5,0,0). (No, you were not expected to list them, just to count them).