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M382D Midterm Exam Solutions
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**Problem 1:** *a) Suppose that $X$ is an $n$-dimensional manifold, $x$ is a point on $X$, and $f^1, \ldots, f^{n-1} : X \to \mathbb{R}$ are smooth functions such that the differentials $df^1_x, \ldots, df^{n-1}_x$ are linearly independent at $x$. Prove there is a function $f^n : X \to \mathbb{R}$ such that $f^1, \ldots, f^n$ is a local coordinate system in a neighborhood of $x$.*

We must find a function $f^n$ such that $df^n_x$ is linearly independent of $df^1_x, \ldots, df^{n-1}_x$, since in that case $d\vec{f}_x$ is an invertible matrix, and by the inverse function theorem $\vec{f}$ is a local diffeomorphism, and the functions $f^1, \ldots, f^n$ serve as local coordinates.

Finding $f^n$ is easy if $X = \mathbb{R}^n$. Just pick a vector $v$ that is linearly independent of $df^1_x, \ldots, df^{n-1}_x$, and let $f^n(x) = v \cdot x$. If $X$ is a general $n$-manifold, with a local coordinates $\phi : y_1, \ldots, y_n \to X$, apply the same construction to the $y$'s. That is, let $f^n$ be a linear function of the $y$'s, obtained by taking the innner product with a vector that is linearly independent of the existing $df$'s, expressed in the $y$ coordinates.

**Problem 2:** *a) Suppose $f : X \to Y$ is a smooth map from a compact manifold $X$ to a connected manifold $Y$. Assume that $df_x$ is invertible for all $x \in X$. Prove that $f$ is surjective.*

Actually, this is worded badly. One has to assume that $X$ is nonempty! Furthermore, the proof is different in dimension 0 from positive dimension.

If $X$ is 0-dimensional, then so is $Y$, and since $Y$ is connected, $Y$ is a single point, so $f$ is onto.

If $dim X > 0$, then the inverse function theorem says that $f$ is a local diffeomorphism. This implies that the image of $f$ is open, since each point in the image has a neighborhood that is diffeomorphic to a neighborhood in $X$. However, $X$ is compact, so $f(X)$ is compact, so $f(X)$ is closed. Since $f(X)$ is nonempty and both open and closed, and since $Y$ is connected, $f(X)$ is all of $Y$.

*b) Find a counterexample if $X$ is not compact.*

Let $f$ be the inclusion of the interval $X = (0, 1)$ in the real line $Y$.

**Problem 3:** *Let $X$ be a smooth manifold and let $f : X \to \mathbb{R}^3$ be a smooth map.*

*a) Is there necessarily a point $z \in \mathbb{R}^3$ such that $f^{-1}(x)$ is a smooth submanifold of $X$?*

By Sard's theorem, almost every point in $\mathbb{R}^3$ is a regular value of $f$, and the preimage of a regular value is a smooth submanifold of $X$.

*b) Is there necessarily a vertical line $\ell$ in $\mathbb{R}^3$ such that $f^{-1}(\ell)$ is a smooth submanifold of $X$?*

Yes. Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection $\pi(x, y, z) = (x, y)$ and let $g = \pi \circ f$. Then by

Sard, almost every point in $\mathbb{R}^2$ is a regular value of $g$. Pick such a regular value $p$, and let $\ell = \pi^{-1}(p)$. Then $f^{-1}(\ell) = g^{-1}(p)$ is a smooth submanifold of $X$.

Note that Sard's theorem does NOT imply that there is a line of regular values of $f$, only that there exists a regular value of $g$, which is all we need to show that $f^{-1}(\ell)$ is a smooth submanifold.

**Problem 4:** *a) Suppose we have the usual situation for intersection theory ($X$ compact, $Z$ closed submanifold of $Y$, and $dim(X) + dim(Z) = dim(Y)$) and that $f : X \to Y$ is homotopic to a constant map. Show that $I_2(f, Z) = 0$.*

Again, we need the dimension of $X$ to be positive. First we show that if $f$ is homotopic to a constant map, then it is homotopic to a constant map *that misses* $Z$. Since $X$ has dimension greater than zero, $Z$ had dimension less than $Y$, so every point $p \in Z$ is in the same path-component of $Y$ as a point $q \notin Z$. If $\gamma(t)$ is a path from $\gamma(0) = p$ to $\gamma(1) = q$, then $F : X \times I \to Y, F(x, t) = \gamma(t)$ is a homotopy from a constant map with image $p$ to a constant map with image $q$.

However, a map that misses $Z$ is automatically transversal to $Z$, and has intersection number zero. Since mod-2 intersection number is a homotopy invariant, our original map $f$ must have $I_2(f, Z) = 0$.

*b) Suppose that $Y = \mathbb{R}^N$, that we have the usual setup for intersection theory, and that $f : X \to Y$ is any smooth map. Show that $I_2(f, Z) = 0$.*

This is a corollary of part (a). Since $\mathbb{R}^N$ is contractible, every map $X \to Y$ is homotopic to the zero map. (For instance, take $F(x, t) = (1 - t)f(x)$.)

**Problem 5:** *Prove that there exists a complex number $z$ such that $z^7 + \cos(|z|^2)(35z^3 + iz^2 - 894) = 0$. Don't handwave! If you claim that two maps are homotopic, show the homotopy explicitly.*

Let $f(z) = z^7 + \cos(|z|^2)(35z^3 + iz^2 - 894) = 0$, and let $u(z) = f(z)/|f(z)|$, except where $f(z) = 0$. Now let $W$ be the closed ball of radius $r_n = \sqrt{(n + (1/2))\pi}$, where $n$ is any non-negative integer. On the boundary of $W$, $f(z)$ is just $z^7$, since $\cos(|z|^2) = 0$, so the degree of $u$, as a map from the circle of radius $r_n$ to $S^1$, is $7 = 1 \pmod 2$. [Actually, I had intended you to show that $f$ was homotopic to $z^7$, and hence had the same winding number, but this trick makes that unnecessary.] By the Extension Theorem, this means that $u$ cannot be extended to a map from all of $W$ to $S^1$, and hence that $f$ must have a zero somewhere on $W$.